8.3 Trig Integrals (part 2)

__example__

$$\int \tan^3 x \, dx$$

two ways to do subs
for this example

$\frac{d}{dx} \tan x = \sec^2 x$

$\frac{d}{dx} \sec x = \sec x \tan x$

just like w/ $\cos x$ and $\sin x$,
we want to somehow bring in
something to make substitution
doable

first way:

$$\int \tan x \cdot \tan^2 x \, dx$$

$\tan^2 x + 1 = \sec^2 x$

$$= \int \tan x \cdot (\sec^2 x - 1) \, dx$$

$$= \underbrace{\int \tan x \cdot \sec^2 x \, dx}_{\substack{u = \tan x \\ du = \sec^2 x \, dx}} - \underbrace{\int \tan x \, dx}_{\int \frac{\sin x}{\cos x} dx}$$

$u = \tan x$
$du = \sec^2 x \, dx$

$\int \frac{\sin x}{\cos x} \, dx$ $\quad U = \cos x$
$dU = -\sin x \, dx$

$$= \dots = \boxed{\frac{1}{2} \tan^2 x + \ln|\cos x| + C}$$

the second way:

$$\int \tan^3 x \, dx$$

$\frac{d}{dx} \tan x = \sec^2 x \qquad \frac{d}{dx} \sec x = \sec x \tan x$

$\tan^2 x + 1 = \sec^2 x$

$$= \int \frac{\tan^3 x}{\sec x} \cdot \sec x \, dx = \int \frac{\tan^2 x}{\sec x} \cdot \tan x \sec x \, dx$$

$$= \int \frac{\sec^2 x - 1}{\sec x} \cdot \sec x \tan x \, dx$$

$u = \sec x$

$du = \sec x \tan x \, dx$

$$= \int \frac{u^2 - 1}{u} \, du = \int \left(u - \frac{1}{u}\right) du = \frac{1}{2} u^2 - \ln|u| + C$$

$$= \boxed{\frac{1}{2} \sec^2 x - \ln|\sec x| + C}$$

they look different

$$\frac{1}{2}(\tan^2 x + 1) - \ln\left|\frac{1}{\cos x}\right| + C$$

$$= \frac{1}{2} \tan^2 x - \ln(\cos^{-1} x) + \frac{1}{2} + C$$

$$= \frac{1}{2} \tan^2 x + \ln(\cos x) + B$$ constant

example

$$\int \frac{1}{\sec x - 1}\,dx$$

another way to handle $\sec x$ and $\tan x$: turn into $\sin x$ and $\cos x$

$$= \int \frac{1}{\frac{1}{\cos x} - 1}\,dx = \int \frac{1}{\frac{1}{\cos x} - 1} \cdot \frac{\cos x}{\cos x}\,dx$$

$$= \int \frac{\cos x}{1 - \cos x}\,dx = \int \frac{\cos x}{1 - \cos x} \cdot \frac{1 + \cos x}{1 + \cos x}\,dx$$

$$= \int \frac{\cos x + \cos^2 x}{1 - \cos^2 x}\,dx = \int \frac{\cos x + \cos^2 x}{\sin^2 x}\,dx$$

$$= \int \frac{\cos x}{\sin^2 x}\,dx + \int \frac{\cos^2 x}{\sin^2 x}\,dx$$

$$= \int \frac{\cos x}{\sin^2 x}\,dx + \int \frac{1 - \sin^2 x}{\sin^2 x}\,dx = \int \frac{\cos x}{\sin^2 x}\,dx + \int \frac{1}{\sin^2 x}\,dx - \int 1\,dx$$

$$= \underbrace{\int \frac{\cos x}{\sin^2 x}\,dx}_{\substack{u = \sin x \\ du = \cos x\,dx}} + \underbrace{\int \csc^2 x\,dx}_{-\cot x} - \underbrace{\int 1\,dx}_{x}$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\vdots$$

$$= \boxed{-\frac{1}{\sin x} - \cot x - x + C}$$

strategy for $\int \tan^m x \sec^n x \, dx$

if <u>n is even and positive</u> $\rightarrow$ save $\sec^2 x$ and use $u = \tan x$ along with identities

$\sec x$ has pos. even power

if <u>m is odd and positive</u> $\rightarrow$ save $\sec x \tan x$ and use $u = \sec x$ along w/ identities

$\tan x$ has pos. odd power

<u>example</u> $\int \tan x \sec^7 x \, dx$

here, $\tan x$ has pos. odd power (1)

save $\sec x \tan x$

$= \int \sec^6 x \cdot \sec x \tan x \, dx$ $\quad u = \sec x \quad du = \sec x \tan x \, dx$

$= \int u^6 \, du = \frac{1}{7} u^7 + c = \boxed{\frac{1}{7} \sec^7 x + c}$

<u>example</u> $\int \tan^5 x \sec^6 x$ $\tan x$ has pos. odd, $\sec x$ has pos. even

→ can use either strategy

try saving $\sec^2 x$, $u = \tan x$

$$\int \tan^5 x \sec^4 x \sec^2 x \, dx$$

↳ $(\sec^2 x)^2 = (\tan^2 x + 1)^2$

$$= \int \tan^5 x (\tan^2 x + 1)^2 \sec^2 x \, dx \qquad u = \tan x$$

$$du = \sec^2 x \, dx$$

$$= \int u^5 (u^2+1)^2 \, du = \int u^5 (u^4 + 2u^2 + 1) \, du$$

$$= \int (u^9 + 2u^7 + u^5) \, du = \frac{1}{10} u^{10} + \frac{1}{4} u^8 + \frac{1}{6} u^6 + C$$

$$= \boxed{\frac{1}{10} \tan^{10} x + \frac{1}{4} \tan^8 x + \frac{1}{6} \tan^6 x + C}$$

$\int \tan^5 x \sec^6 x \, dx$ now try saving $\sec x \tan x$ and $u = \sec x$

$= \int \tan^4 x \sec^5 x \sec x \tan x \, dx$

$\hookrightarrow (\tan^2 x)^2 = (\sec^2 x - 1)^2$

$= \int (\sec^2 x - 1)^2 \sec^5 x \sec x \tan x \, dx$

$u = \sec x$

$du = \sec x \tan x \, dx$

$= \int (u^2 - 1)^2 u^5 \, du = \int (u^4 - 2u^2 + 1) u^5 \, du$

$= \int (u^9 - 2u^7 + u^5) \, du = \frac{1}{10} u^{10} - \frac{1}{4} u^8 + \frac{1}{6} u^6 + C$

$= \boxed{\frac{1}{10} \sec^{10} x - \frac{1}{4} \sec^8 x + \frac{1}{6} \sec^6 x + C}$