8.4 Trig Subs (part 2)

example Find the length of $y = \sqrt{1-x^2}$ from $x = 0$ to $x = \frac{1}{\sqrt{2}}$

$$L = \int_a^b \sqrt{1 + (y')^2}\, dx$$

$$y = (1-x^2)^{1/2}$$

$$y' = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1-x^2}}$$

$$1 + (y')^2 = 1 + \left(\frac{-x}{\sqrt{1-x^2}}\right)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}$$

$$L = \int_0^{\frac{1}{\sqrt{2}}} \sqrt{\frac{1}{1-x^2}}\, dx = \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}}\, dx$$

now knowing trig subs
we can evaluate this

triangle with sides $\sqrt{1-x^2}$, 1, x

hypotenuse : 1

adjacent : $\sqrt{1-x^2}$ (part containing a constant)

relate x and θ using simplest sides :

$\sin\theta = \frac{x}{1}$

$x = \sin\theta \rightarrow \theta = \sin^{-1}(x)$

$dx = \cos\theta \, d\theta$

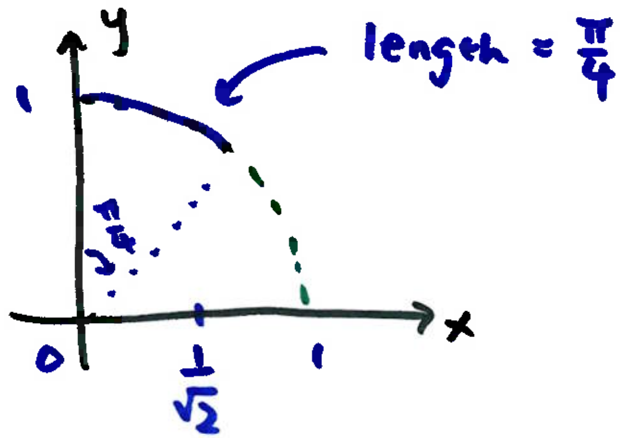
upper limit : $x = \frac{1}{\sqrt{2}} \rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

lower limit : $x = 0 \rightarrow \theta = \sin^{-1}(0) = 0$

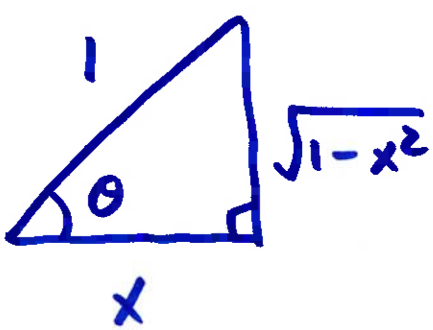
$$\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}} dx$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{\underbrace{1-\sin^2\theta}_{\cos^2\theta}}} \cos\theta \, d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{\cos\theta} \cos\theta \, d\theta = \int_0^{\frac{\pi}{4}} d\theta = \theta \Big|_0^{\pi/4} = \boxed{\frac{\pi}{4}}$$

geometrically, this was part of the circumference of a circle radius 1



we <u>cannot</u> have other hypotenuse, both adj and opp can be swapped



relate x and θ : $\cos\theta = \frac{x}{1}$

$x = \cos\theta \rightarrow \theta = \cos^{-1}(x)$

$dx = -\sin\theta \, d\theta$

$x = \frac{1}{\sqrt{2}} \rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$x = 0 \rightarrow \theta = \cos^{-1}(0) = \frac{\pi}{2}$

$$\int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1-x^2}} dx = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{1}{\sqrt{1-\cos^2\theta}} \cdot -\sin\theta \, d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{1}{\sin\theta} \cdot -\sin\theta \, d\theta = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} -d\theta = -\theta \Big|_{\frac{\pi}{2}}^{\frac{\pi}{4}} = -\frac{\pi}{4} + \frac{\pi}{2} = \boxed{\frac{\pi}{4}}$$

the rule of thumb of putting constant or constant-containing part as adjacent limits the subs to 3: $\sin\theta$, $\sec\theta$, $\tan\theta$

swapping adj and opp brings in additional 3: $\cos\theta$, $\csc\theta$, $\cot\theta$

example

$$\int \frac{1}{x^2-6x+45}\,dx = \int \frac{1}{\left(\sqrt{x^2-6x+45}\right)^2}\,dx$$

notice the part under radical doesn't look like a sum or difference of squares → useful for building triangles

complete the square : $x^2-6x+45$

$$= x^2-6x+\left(\frac{-6}{2}\right)^2+45-\left(\frac{-6}{2}\right)^2$$

$$= x^2-6x+9+45-9$$

$$= (x-3)^2+6^2$$

$$\int \frac{1}{\left(\sqrt{(x-3)^2+6^2}\right)^2}\,dx$$

$$\int \frac{1}{\left(\sqrt{(x-3)^2+6^2}\right)^2}\,dx$$

$u = x-3$

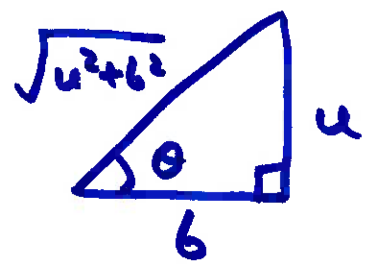
$du = dx$

$$\int \frac{1}{\left(\sqrt{u^2+6^2}\right)^2}\,du$$

triangle w/ sides $\sqrt{u^2+6^2}$, u, 6

hypotenuse: $\sqrt{u^2+6^2}$

adjacent: 6



$\tan\theta = \frac{u}{6} \rightarrow u = 6\tan\theta$

$du = 6\sec^2\theta\,d\theta$

$$\int \frac{1}{\Big(\sqrt{\underbrace{36\tan^2\theta+36}_{\substack{36(\tan^2\theta+1)\\ 36\sec^2\theta}}}\Big)^2}\cdot 6\sec^2\theta\,d\theta = \int \frac{1}{36\sec^2\theta}\cdot 6\sec^2\theta\,d\theta$$

$$= \int \frac{1}{6}\, d\theta = \frac{1}{6}\theta + C$$

now back to u, back to x

from $u = 6\tan\theta \rightarrow \theta = \tan^{-1}\left(\frac{u}{6}\right) = \tan^{-1}\left(\frac{x-3}{6}\right)$

$$= \boxed{\frac{1}{6}\tan^{-1}\left(\frac{x-3}{6}\right) + C}$$

__example__

$$\int \frac{1}{\sqrt{(x-5)(1-x)}}\,dx$$

make what's under radical a sum or difference of squares

$$(x-5)(1-x) = -x^2+6x-5$$

$$= -(x^2-6x+5)$$ complete square inside ( )

$$= -(x^2-6x+9+5-9)$$

$$= -\left[(x^2-6x+9)-4\right] = -\left[(x-3)^2-2^2\right]$$

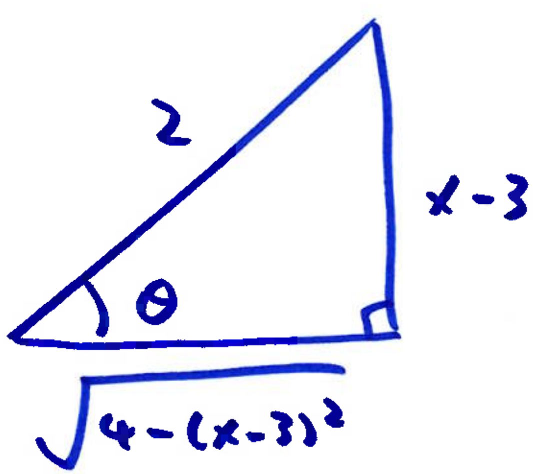
$$= 2^2-(x-3)^2$$

$$\int \frac{1}{\sqrt{2^2-(x-3)^2}}\,dx$$

triangle w/ sides $\sqrt{4-(x-3)^2}$, 2, $x-3$

hypotenuse : 2

adjacent : $\sqrt{4-(x-3)^2}$



$$\sin\theta = \frac{x-3}{2}$$

$$x-3 = 2\sin\theta$$

$$x = 2\sin\theta + 3$$

$$dx = 2\cos\theta \, d\theta$$

$$\int \frac{1}{\sqrt{4-(\underbrace{x-3}_{2\sin\theta})^2}}\,dx = \int \frac{1}{\sqrt{\underbrace{4-4\sin^2\theta}_{\substack{4(1-\sin^2\theta)\\ 4\cos^2\theta}}}} \cdot 2\cos\theta\,d\theta = \int \frac{1}{\sqrt{4\cos^2\theta}} \cdot 2\cos\theta\,d\theta$$

$$= \int d\theta = \theta + C$$

from $\sin\theta = \frac{x-3}{2}$ we got $\theta = \sin^{-1}\left(\frac{x-3}{2}\right)$

$$= \boxed{\sin^{-1}\left(\frac{x-3}{2}\right) + C}$$