8.5 Partial Fractions (part 2)

last time : linear factors, some may be repeated

$$\frac{1}{(x)(x-2)} = \frac{A}{x} + \frac{B}{x-2} \qquad \text{find } A, B$$

$$\frac{1}{(x)(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \qquad \text{find } A, B, C$$

numerator must have lower degree than denominator

3rd degree

$$\frac{26x^{3} - 52x^2 + 2}{x^{2} - 2x}$$

must reduce before expansion

2nd degree

last time: rearrangement

this time: long division

$$\frac{26x^3 - 52x^2 + 2}{x^2 - 2x}$$

x^2-2x goes $26x$ times into $26x^3-52x^2+2$

$$\begin{array}{r} 26x \\ x^2-2x \,\big)\, 26x^3 - 52x^2 + 2 \\ -(26x^3 - 52x^2 \quad) \\ \hline 0 + \cancel{52x^2}\, 0 + 2 \end{array}$$

remainder

0th degree

$$= 26x + \frac{2}{x^2-2x}$$

2nd degree

$$= 26x + \underbrace{\frac{2}{x(x-2)}}_{\frac{A}{x} + \frac{B}{x-2}}$$

$$\frac{2}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$$2 = A(x-2) + B(x)$$

$$\int 26x + \left(\frac{A}{x} + \frac{B}{x-2}\right) dx$$

now quadratic factors

reducible : for example, $x^2-2x = (x)(x-2)$ product of linear factors

irreducible : cannot be factored into linear factors

for example, x^2+4

$$\frac{2x^2-x+4}{x^3+4x} = \frac{2x^2-x+4}{(x)(x^2+4)}$$

linear

irreducible quadratic

$$= \underbrace{\frac{A}{x}}_{} + \underbrace{\frac{Bx+C}{x^2+4}}_{2\text{nd degree}}$$

linear (1st deg) for irreducible quadratic

for the linear factor x numerator is constant (0th degree)

1st degree

$\boxed{\text{numerator is one degree lower than denominator}}$

example

$$\int \frac{x^2+x+2}{(x+1)(x^2+1)}\,dx$$

degree check: numerator (2nd) < denom. (3rd)

ok

expansion: $\dfrac{x^2+x+2}{(x+1)(x^2+1)} = \dfrac{A}{x+1} + \dfrac{Bx+C}{x^2+1}$ find A, B, C

linear

irreducible quadratic

multiply by $(x+1)(x^2+1)$

$$x^2+x+2 = A(x^2+1) + (Bx+C)(x+1)$$

$$= Ax^2 + A + Bx^2 + Bx + Cx + C$$

$$x^2+x+2 = \underline{(A+B)}\,x^2 + \underline{(B+C)}\,x + \underline{(A+C)}$$

$A+B = 1$ — ①

$B+C = 1$ — ②

$A+C = 2$ — ③

from ① $B = 1 - A$ sub into ②

② : $B + C = 1$

$1 - A + C = 1$

$C = A$ sub into ③

③ : $A + C = 2$

$A + A = 2 \rightarrow \boxed{A = 1}$ $\boxed{C = 1}$ $\boxed{B = 0}$

$$\int \frac{x^2 + x + 2}{(x+1)(x^2+1)}\,dx = \int \left(\frac{A}{x+1} + \frac{Bx + C}{x^2+1} \right) dx$$

$$= \int \left(\frac{1}{x+1} + \frac{1}{x^2+1} \right) dx = \int \frac{1}{x+1}\,dx + \int \frac{1}{x^2+1}\,dx$$

trig subs

$$= \boxed{\ln|x+1| + \tan^{-1}(x) + D}$$

repeated irreducible quadratic factors are handled just like how we handle repeated linear factors

for example, $\dfrac{1}{(x)(x^2+1)^2} = \dfrac{1}{\underset{\text{linear}}{(x)}\underbrace{(x^2+1)(x^2+1)}_{\text{irreducible and repeated}}}$

$$\frac{1}{(x)(x^2+1)(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$ find A, B, C, D, E

multiply by $(x)(x^2+1)(x^2+1)$

$$1 = A(x^2+1)(x^2+1) + (Bx+C)(x)(x^2+1) + (Dx+E)(x)$$

$\vdots$ multiply out, collect by power

$$1 = (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

$$0x^4 + 0x^3 + 0x^2 + 0x + 1$$

$$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

right away we see $\boxed{C=0}$, $\boxed{A=1}$

then $A+B=0 \rightarrow B=-A \rightarrow \boxed{B=-1}$

then $C+E=0 \rightarrow E=-C \rightarrow \boxed{E=0}$

$2A+B+D=0$

$2(1)+(-1)+D=0 \rightarrow \boxed{D=-1}$

$$\int \frac{1}{(x)(x^2+1)(x^2+1)}\,dx$$

$$= \int \left(\frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} \right) dx$$

$$= \int \left(\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2} \right) dx = \underbrace{\int \frac{1}{x}\,dx}_{\text{easy}} - \int \frac{x}{x^2+1}\,dx - \int \frac{x}{(x^2+1)^2}\,dx$$

sub $u = x^2+1$

$\vdots$

$$= \boxed{\ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2}\frac{1}{x^2+1} + F}$$

write the <u>form</u> of expansion

$$\frac{2x^3+5x-10}{x^2(x^2+4)^3(x^2-9)^2} = \frac{2x^3+5x-10}{(x)(x)(x^2+4)(x^2+4)(x^2+4)(x+3)(x+3)(x-3)(x-3)}$$

$(x+3)^2(x-3)^2$

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+4} + \frac{Ex+F}{(x^2+4)^2} + \frac{Gx+H}{(x^2+4)^3} + \frac{I}{x+3} + \frac{J}{(x+3)^2} + \frac{K}{x-3} + \frac{L}{(x-3)^2}$$