

10.2 Sequences

this section is not on exam 2

Sequence: list of numbers in some order

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \dots\}$$

sequence converges if $\lim_{n \rightarrow \infty} a_n$ exists

for example, $\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots, \frac{99,999}{100,000}, \dots \right\}$

the terms appear to approach 1

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$$

so $\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty}$ converges to 1

example

$\left\{ n^{1/3} \right\}_{n=1}^{\infty}$ converges?

$$= \left\{ 1^{1/3}, 2^{1/3}, 3^{1/3}, \dots, 13^{1/3}, \dots \right\}$$

the terms appear to initially get bigger

does the sequence converge?

listing numbers does not seem to answer that question

$$\text{try } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} n^{1/3} \rightarrow \infty \quad \text{indeterminate}$$

limit could be anything

L'Hospital's Rule can give us the limit

let $y = n^{1/3}$ \rightarrow we want $\lim_{n \rightarrow \infty} y$

$$\ln y = \ln n^{1/3} = \frac{1}{3} \ln(n) = \frac{\ln(n)}{3} \rightarrow \frac{\infty}{\infty} \quad \text{as } n \rightarrow \infty$$

we will use L'Hospital's Rule on this

$$\lim_{n \rightarrow \infty} \frac{13 \ln(n)}{n}$$

$$= \lim_{n \rightarrow \infty} \frac{13 \cdot \frac{1}{n}}{1} = \lim_{n \rightarrow \infty} \frac{13}{n} = 0$$

this means $\lim_{n \rightarrow \infty} \ln y = 0$ but we want $\lim_{n \rightarrow \infty} y$ where $y = n^{13/n}$

remember $e^{\ln y} = y$

so, $\lim_{n \rightarrow \infty} \ln y = 0$ becomes

$\lim_{n \rightarrow \infty} e^{\ln y} = e^0 = \boxed{1}$ this is the limit of $\{n^{13/n}\}_{n \in \mathbb{N}}$

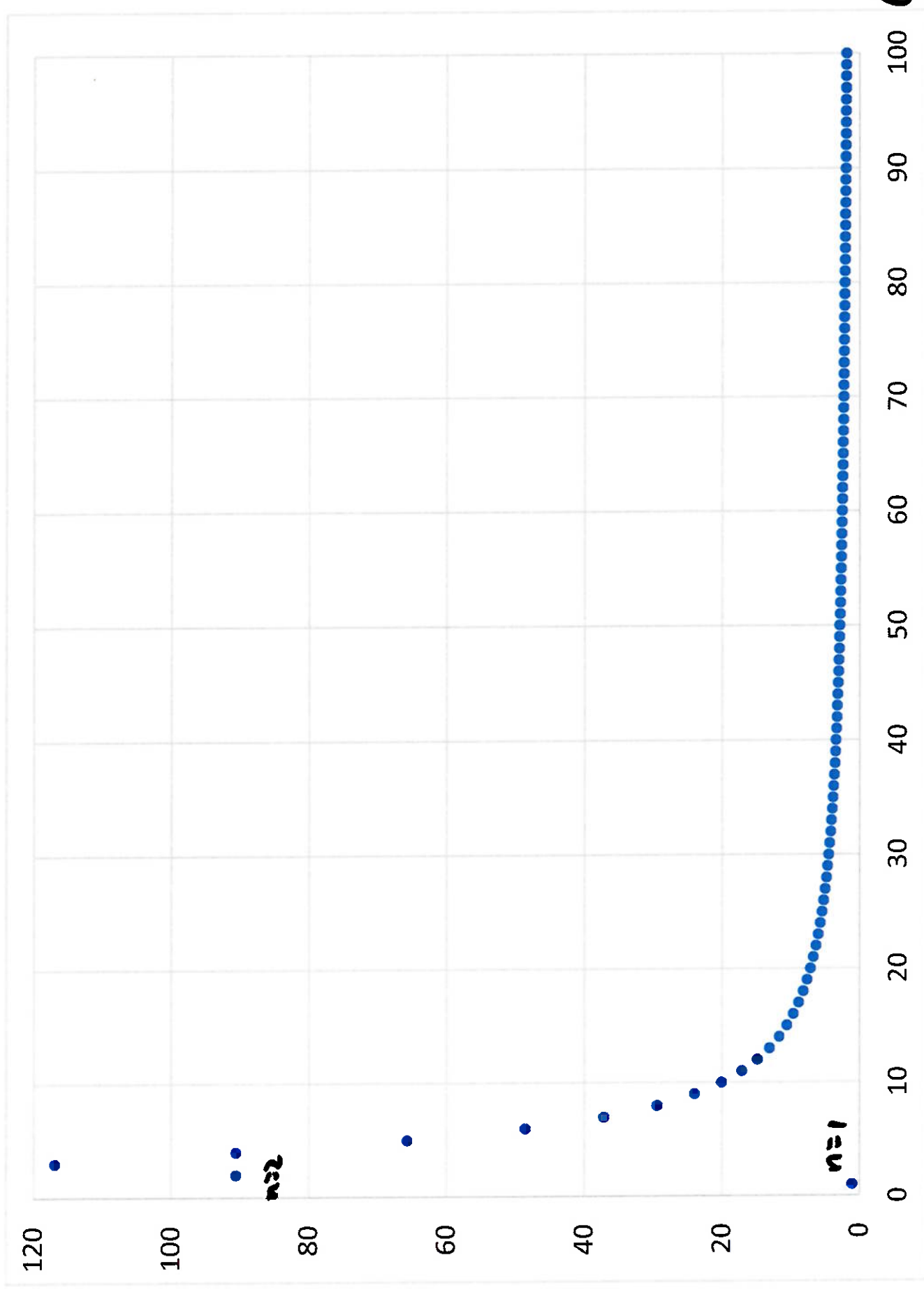
$\{1, 2^{13/2}, 3^{13/3}, \dots, 1,000,000^{13/1,000,000}, \dots\}$

wait long enough, very close to 1

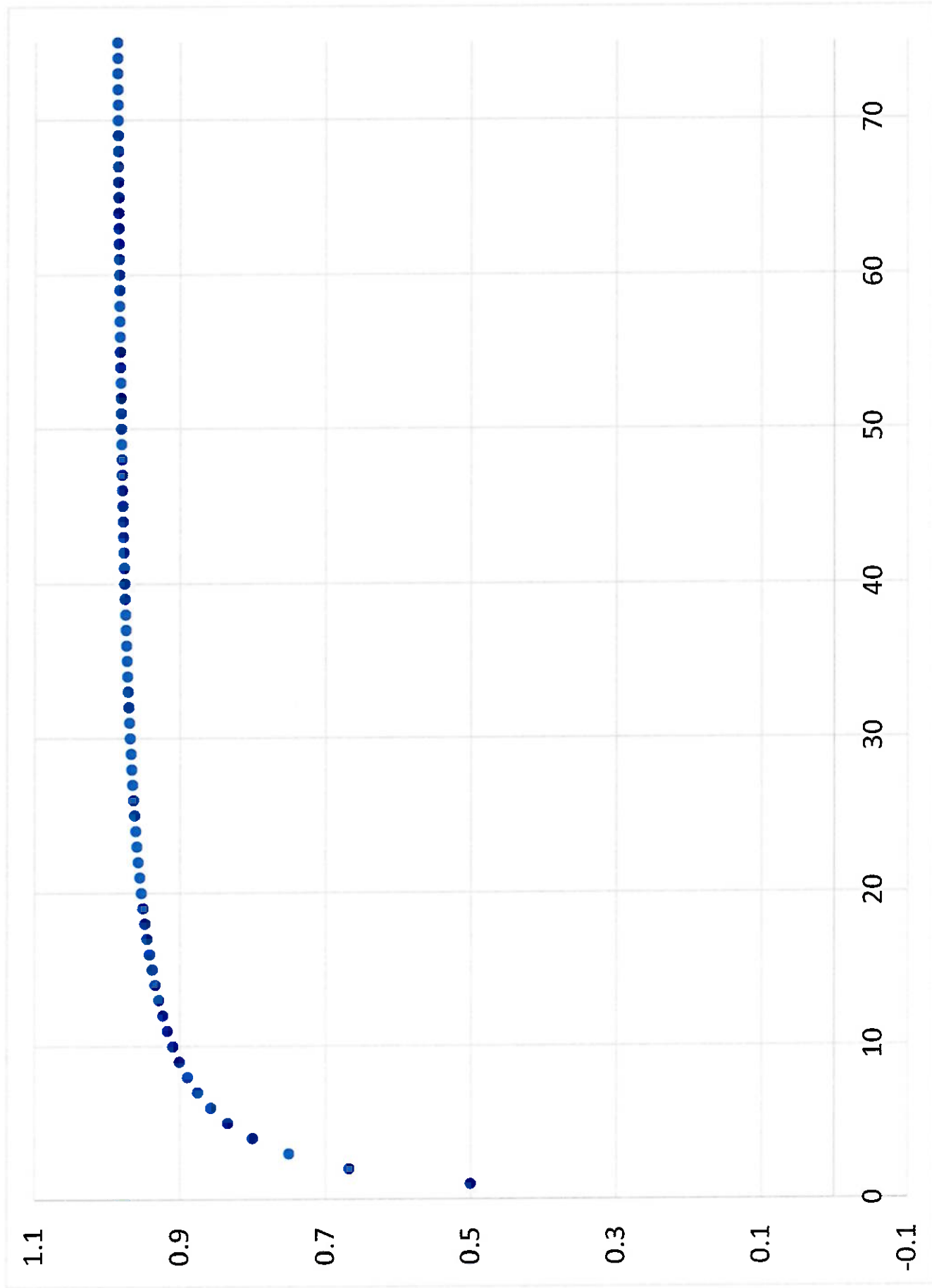
$$a_n = n^{13/n}$$

(a_n)

(n)



$$\left\{ \frac{r}{n+1} \right\}$$



$\{n^{13/n}\}$ increases initially, then decreases and settles down around 1

whereas $\left\{\frac{n}{n+1}\right\}$ always increases

if a sequence either always increases or always decreases (for all n)

then the sequence is said to be monotonic

$\{1, 2, 3, 4, 5, \dots\}$ is monotonic

$\{-1, -2, -3, -5, -7, -9, -13, \dots\}$ is monotonic

$\left\{-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \dots\right\}$ is not monotonic

$$\left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots \right\}$$

Smallest term in this sequence
increases to approach 1

this sequence is bounded because no all terms are no smaller than something ($\frac{1}{2}$) and no bigger than something else (1)

if a sequence is bounded and monotonic, then it converges

An important sequence is the geometric sequence $\rightarrow \{r^n\}$ r is constant

$$\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots \right\}$$

$$= \left\{ \left(\frac{1}{2}\right)^1, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \left(\frac{1}{2}\right)^4, \dots \right\}_{n=1}^{\infty} \text{ converges to } 0$$

$$\hookrightarrow r = \frac{1}{2}$$

Common ratio between
succeeding terms

$$\left\{ -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \dots \right\}_{n=1}^{\infty} \text{ converges to } 0$$

$$\left\{ 2, 4, 8, 16, 32, 64, \dots \right\}_{n=1}^{\infty} \text{ diverges}$$

the ratio r appears to determine convergence

the geometric sequence $\{r^n\}$ converges if $-1 < r \leq 1$

why is -1 included by -1 is not?

$$\{r^n\}_{n=1}^{\infty}$$

if $r=1$: $\{1, 1, 1, 1, 1, \dots\}$ converges to 1

if $r=-1$: $\{-1, 1, -1, 1, -1, 1, \dots\}$ no limit, diverges

factorial (!) shows up a lot in sequences and series

$$n! = (n)(n-1)(n-2) \dots (1)$$

$$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

example $\left\{ \frac{n}{n!} \right\}_{n=1}^{\infty}$

$$= \left\{ 1, \frac{2}{2!}, \frac{3}{3!}, \frac{4}{4!}, \dots \right\}$$

$$= \left\{ 1, 1, \frac{1}{2}, \frac{1}{6}, \dots \right\}$$

converges? check $\lim_{n \rightarrow \infty} \frac{n}{n!}$

$n \rightarrow \infty$ numerator is large
denominator is

very LARGE

limit appears to be zero

let's check property: $\lim_{n \rightarrow \infty} \frac{n}{n!} = \lim_{n \rightarrow \infty} \frac{\cancel{n}}{\cancel{n}(n-1)(n-2) \dots (1)}$

$$= \lim_{n \rightarrow \infty} \frac{1}{(n-1)(n-2) \dots (1)} = 0 \text{ so, } \left\{ \frac{n}{n!} \right\} \text{ converges to } 0$$

example $\left\{ \frac{e^n}{n!} \right\}_{n=1}^{\infty}$

$$= \left\{ \frac{e^1}{1!}, \frac{e^2}{2!}, \frac{e^3}{3!}, \dots \right\}$$

initially, e^n dominates $n!$

as n becomes large, for example

$$n=10 : e^{10} \approx 22,026$$

$$10! \approx 3,628,800$$

$$n=100 : e^{100} \approx 3 \times 10^{43}$$

$$100! \approx 9 \times 10^{157}$$

$n \rightarrow \infty$, denominator wins, limit appears to be zero

can we be more sure?

$$\frac{e^n}{n!} = \frac{e^n}{(n)(n-1)(n-2)\dots(1)}$$

as n becomes large, the denominator is polynomial w/ leading term n^n
which dominates

so, as $n \rightarrow \infty$, $\frac{e^n}{n!} \approx \frac{e^n}{n^n} = \left(\frac{e}{n}\right)^n$ looks like r^n
with $r \rightarrow 0$ as $n \rightarrow \infty$
and since $-1 < r \leq 1$
this behaves like a
geo. seq. as $n \rightarrow \infty$
therefore converges.