

10.7 Ratio and Root Tests

Given $\sum_{k=1}^{\infty} a_k$, if $\sum_{k=1}^{\infty} |a_k|$ converges, then series $\sum_{k=1}^{\infty} a_k$

converges absolutely

e.g. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges

so, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges absolutely

if a series converges absolutely, then it converges

if $\sum_{k=1}^{\infty} a_k$ converges, but $\sum_{k=1}^{\infty} |a_k|$ does not, then $\sum_{k=1}^{\infty} a_k$

converges conditionally

e.g. $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges but $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

Ratio and Root Tests tell us if a series converges absolutely
(and therefore converges)

Ratio Test

Given $\sum_{k=1}^{\infty} a_k$, the series converges absolutely (and therefore converges)

if $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| < 1$

series diverges if $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| > 1$

test is inconclusive if $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = 1$

why? if $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = r$, then this says if k is large,

$|a_{k+1}| \approx r |a_k| \rightarrow$ like a geometric series which
converges if $|r| < 1$

this is why we need $\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| < 1$

Example

$$\sum_{k=1}^{\infty} \underbrace{k \left(\frac{1}{4}\right)^k}_{a_k}$$

$$a_k = k \left(\frac{1}{4}\right)^k$$

$$a_{k+1} = (k+1) \left(\frac{1}{4}\right)^{k+1}$$

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{(k+1) \left(\frac{1}{4}\right)^{k+1}}{k \left(\frac{1}{4}\right)^k} \right|$$

$$= \lim_{k \rightarrow \infty} \left| \frac{k+1}{k} \cdot \underbrace{\left(\frac{1}{4}\right)}_1 \right| = \left| \frac{1}{4} \right| = \frac{1}{4} < 1, \text{ so this series}$$

converges (absolutely)

means the tail of this series ($k \rightarrow \infty$)
behaves like a geo. series w/ $r = \frac{1}{4}$

Ratio test handles factorials well

Example $\sum_{k=1}^{\infty} \frac{k!}{(2k+6)!} a_k$

$$a_{k+1} = \frac{(k+1)!}{(2(k+1)+6)!} = \frac{(k+1)!}{(2k+8)!}$$

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{\frac{(k+1)!}{(2k+8)!}}{\frac{k!}{(2k+6)!}} \right| = \lim_{k \rightarrow \infty} \left| \frac{(k+1)!}{(2k+8)!} \cdot \frac{(2k+6)!}{k!} \right|$$

$$= \lim_{k \rightarrow \infty} \left| \frac{(k+1)!}{k!} \cdot \frac{(2k+6)!}{(2k+8)!} \right|$$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
 $k! = k \cdot (k-1) \cdot (k-2) \cdots (1)$

$$= \lim_{k \rightarrow \infty} \left| \frac{\cancel{(k+1)} \cancel{(k)} \cancel{(k-1)} \cancel{(k-2)} \cdots \cancel{(1)}}{\cancel{(k)} \cancel{(k-1)} \cancel{(k-2)} \cdots \cancel{(1)}} \cdot \frac{\cancel{(2k+6)} \cancel{(2k+5)} \cdots \cancel{(1)}}{\cancel{(2k+8)} \cancel{(2k+7)} \cancel{(2k+6)} \cancel{(2k+5)} \cdots \cancel{(1)}} \right|$$

$$= \lim_{k \rightarrow \infty} \left| \frac{k+1}{(2k+8)(2k+7)} \right| = 0 < 1 \quad \text{so series converges}$$

example

For what values of x does

$$\sum_{k=1}^{\infty} \frac{5x^k}{4k} \text{ converge?}$$

$$= \frac{5}{4}x + \frac{5}{8}x^2 + \frac{5}{12}x^3 + \dots$$

ratio test

$$a_k = \frac{5x^k}{4k} \quad a_{k+1} = \frac{5x^{k+1}}{4(k+1)}$$

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{\frac{5x^{k+1}}{4(k+1)}}{\frac{5x^k}{4k}} \right| = \lim_{k \rightarrow \infty} \left| \frac{5x^{k+1}}{4k+4} \cdot \frac{4k}{5x^k} \right|$$

$$= \lim_{k \rightarrow \infty} \left| \frac{4k}{4k+4} \cdot \frac{5x^{k+1}}{5x^k} \right| = \lim_{k \rightarrow \infty} \left| \frac{k}{k+1} \cdot x \right| = |x| \text{ we want } < 1$$

$$\text{so, } |x| < 1 \rightarrow -1 < x < 1$$

but notice at $x = -1$ and $x = 1$, the ratio is 1

\rightarrow inconclusive

We need to test $x = -1$ and $x = 1$ separately

$$\sum_{k=1}^{\infty} \frac{5x^k}{4k}$$

when $x = -1$, $\sum_{k=1}^{\infty} \frac{5}{4} \frac{(-1)^k}{k} \rightarrow$ converges

when $x = 1$, $\sum_{k=1}^{\infty} \frac{5}{4k} = \sum_{k=1}^{\infty} \frac{5}{4} \cdot \frac{1}{k} \rightarrow$ diverges (harmonic
or p-series w/ $p=1$)

so series converges on

$$\boxed{-1 \leq x < 1}$$

Root Test

Given $\sum_{k=1}^{\infty} a_k$, it converges absolutely if

$$\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} < 1$$

$$\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} < 1$$

Series diverges if $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} > 1$

inconclusive if $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = 1$

why? if $\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = r$

then for large k , $\sqrt[k]{|a_k|} \approx r$ so $|a_k| \approx r^k$

so like a geo. series w/
ratio r

Root test is good when a_k has some kind of k in exponent

example

$$\sum_{k=1}^{\infty} \frac{(k+1)^k}{k^2 k}$$

$$a_k = \frac{(k+1)^k}{k^2 k} = \frac{(k+1)^k}{(k^2)^k} = \left(\frac{k+1}{k^2}\right)^k$$

root test

$$\lim_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \lim_{k \rightarrow \infty} \sqrt[k]{\left(\frac{k+1}{k^2}\right)^k} = \lim_{k \rightarrow \infty} \frac{k+1}{k^2} = 0 < 1$$

so converges

Ratio Test is good with lots of series, so often it is a good first test to try.

Good w/ factorials, things you don't want to integrate, things you don't want to compare to other things

not good w/ things that look like p-series

example
$$\sum_{k=1}^{\infty} \frac{k}{k^{2+1}}$$

$$\lim_{k \rightarrow \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{\frac{k+1}{(k+1)^{2+1}}}{\frac{k}{k^{2+1}}} \right|$$

$$= \lim_{k \rightarrow \infty} \left| \underbrace{\frac{k+1}{k}}_1 \cdot \underbrace{\frac{k^{2+1}}{(k+1)^{2+1}}}_1 \right| = 1 \text{ inconclusive}$$

but a comparison tells us the result quickly
 $\frac{k}{k^{2+1}} \approx \frac{k}{k^2} \approx \frac{1}{k}$ as $k \rightarrow \infty$, so $\sum_{k=1}^{\infty} \frac{k}{k^{2+1}}$ is like $\sum \frac{1}{k}$ (diverges)