10.8 Choosing a Convergence Test

Summary of Tests

Divergence Test: $\lim_{k\to\infty} a_k = 0 \rightarrow$ series might converge, test more

$\lim_{k\to\infty} a_k \neq 0 \rightarrow$ series diverges

Integral Test: $\sum_{k=1}^{\infty} a_k$ converges if $\int_1^{\infty} a_k(x)\,dx$ converges

p-series Test: $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$

Geometric Series: $\sum_{k=0}^{\infty} ar^k$ converges if $|r| < 1$

Sum $= \frac{a}{1-r}$ ← first term

Direct Comparison Test: if $a_k \leq$ terms of convergent series then $\sum a_k$ converges

if $a_k \geq$ terms of divergent series then $\sum a_k$ diverges

often compare to p-series or geometric

Limit Comparison Test: $\sum a_k$ unknown $\quad \sum b_k$ known (typically a p-series or geo. series)

$$\lim_{k\to\infty} \frac{a_k}{b_k} = c \quad \text{if } 0 < c < \infty \text{ then } \sum a_k \text{ and } \sum b_k \text{ BOTH converge or BOTH diverge}$$

Alternating Series Test: $\sum_{k=1}^{\infty} (-1)^{k+1} a_k$ converges if $\lim_{k\to\infty} a_k = 0$

AND a_k is eventually non increasing

Ratio Test:

$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| < 1$ converges

$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| > 1$ diverges

$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = 1$ or $\cancel{\infty}$ DNE inconclusive, test more (mistake in lecture + video)

Root Test:

$\lim_{k\to\infty} \sqrt[k]{|a_k|} < 1$ converges

$\lim_{k\to\infty} \sqrt[k]{|a_k|} > 1$ diverges

$\lim_{k\to\infty} \sqrt[k]{|a_k|} = 1$ or $\cancel{\infty}$ DNE inconclusive (mistake in lecture + video)

example $\sum_{k=1}^{\infty} \frac{11k^5 - 9k^3 + 5k + 10}{12k^5 + k^2 - k + 110}$

always do the Div Test first.

$\lim_{k\to\infty} a_k \neq 0$ diverges

example $\sum_{k=1}^{\infty} \left(1 - \frac{1}{k}\right)^k$

fails divergence test: $\lim_{k\to\infty} \left(1 - \frac{1}{k}\right)^k = e^{-1} \neq 0$

diverges

example $\sum_{k=1}^{\infty} \frac{1 + \sin 9k}{k^2}$

passes div. test? yes, test more

comparison is good here, because the terms "look like" $\frac{1}{k^2}$

$-1 \leq \sin 9k \leq 1$

$0 \leq 1 + \sin 9k \leq 2$

so $0 \le \frac{1+\sin 9k}{k^2} \le \frac{2}{k^2}$

so the terms of this series are less than or equal to those of a convergent series ($\sum \frac{2}{k^2}$)

therefore $\sum_{k=1}^{\infty} \frac{1+\sin 9k}{k^2} \le \sum_{k=1}^{\infty} \frac{2}{k^2} = C$ (convergent)

therefore $\sum_{k=1}^{\infty} \frac{1+\sin 9k}{k^2} \le C$ so converges

try limit comparison

compare to $\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} b_k$

$$\lim_{k\to\infty} \left\{ \frac{a_k}{b_k} \right\} = \lim_{k\to\infty} \frac{\frac{1+\sin 9k}{k^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} 1+\sin 9k \quad \text{DNE}$$

this is not easy to conclude

Direct Comp. is better here

example $\displaystyle\sum_{k=2}^{\infty} \frac{5}{k(\ln k)^7}$

passes div. test? yes, test more

integral test is good because $\dfrac{5}{x(\ln x)^7}$ can be integrated with $u = \ln x$, $du = \dfrac{1}{x}$

$$\int_2^{\infty} \frac{5}{x(\ln x)^7}\,dx \text{ converges?}$$

$$= \lim_{b\to\infty} \int_2^{b} \frac{5}{x(\ln x)^7}\,dx \qquad u = \ln x, \quad du = \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{5}{u^7}\,du \neq \infty \qquad \text{so } \sum_{k=2}^{\infty} \frac{5}{k(\ln k)^7} \text{ converges}$$

converges since $7 > 1$

comparison? maybe to $\frac{5}{k^8}$?

$$\frac{5}{k(\ln k)^7} \qquad \ln k < k$$

so $\frac{5}{k(\underbrace{\ln k}_{\text{smaller than } k})^7} \geq \frac{5}{k^8}$

so direct comp. does not work, but a limit comp to $\sum \frac{5}{k^8}$ might work

<u>example</u> $\displaystyle\sum_{k=1}^{\infty} \left(\sqrt{16k^4+1} - 4k^2\right)$

divergence test

$$\lim_{k\to\infty} \left(\underbrace{\sqrt{16k^4+1}}_{\infty} - \underbrace{4k^2}_{\infty}\right) = 0 \ ?$$

$\infty - \infty = ?$

we can't conclude at least in its current form

$$\frac{\sqrt{16k^4+1} - 4k^2}{1} \cdot \frac{\sqrt{16k^4+1} + 4k^2}{\sqrt{16k^4+1} + 4k^2} = \frac{\cancel{16k^4} + 1 - \cancel{16k^4}}{\sqrt{16k^4+1} + 4k^2}$$

$$= \frac{1}{\sqrt{16k^4+1} + 4k^2}$$

$\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{16k^4+1} + 4k^2}$ passes div. test

$\dfrac{1}{\sqrt{16k^4+1}+4k^2}$ when k is large. looks like $\dfrac{1}{\sqrt{16k^4}+4k^2} = \dfrac{1}{8k^2}$

so comparison or limit comparison should work well