11.1 Approx. Functions w/ Polynomials (part 2)

Taylor series of $f(x)$ near $x=a$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

if we chop it off after k, we get the k-th order Taylor Polynomial

for example, $k=2$,

$$P_2 = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 \approx f(x)$$ but NOT exactly equal

the terms we throw away $(k \geq 3)$ form the Remainder

here, the remainder $(k \geq 3)$

remainder $\rightarrow$ $$R_2 = \underbrace{\frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots}_{\text{itself an infinite series which converges to one term}}$$

$\uparrow$ after 2nd order

the remainder converges to

Taylor's Remainder Theorem

$$\boxed{R_k = \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1}}$$

where $a < c < x$

so, in general, approximation of $f(x)$ using k^{th} order Taylor Polynomial

$$f(x) = \underbrace{P_k(x)}_{k^{th}\text{ order Taylor Polynomial}} + \underbrace{R_k(x)}_{\text{remainder}}$$

c is usually not easy to find, so instead of finding it, we usually try to put a bound on $|R_k|$ and use it to estimate error the absolute error is

$$\left| \underset{\text{true}}{f(x)} - \underset{k^{th}\text{ order Taylor polynomial}}{P_k(x)} \right| = |R_k| = \left| \frac{f^{(k+1)}(c)}{(k+1)!}(x-a)^{k+1} \right|$$

we don't find c but instead we find the max $|f^{(k+1)}(x)|$

and call it M

then the |error| is no more than $\left| \frac{M}{(k+1)!}(x-a)^{k+1} \right|$

<u>example</u>

$f(x) = e^{-2x}$ $a = 0$ $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

approx using P_2 and estimate the error.

$f(x) = e^{-2x}$ $f(a) = f(0) = e^0 = 1 = (-2)^0$

$f'(x) = -2e^{-2x}$ $f'(a) = f'(0) = -2e^0 = -2 = (-2)^1$

$f''(x) = 4e^{-2x}$ $f''(0) = 4 = (-2)^2$

$f'''(x) = -8e^{-2x}$ $f'''(0) = -8 = (-2)^3$

$\vdots$

$f^{(k)}(0) = (-2)^k$

Taylor series

$$e^{-2x} = 1 - 2x + \frac{(-2)^2}{2!}x^2 + \frac{(-2)^3}{3!}x^3 + \frac{(-2)^4}{4!}x^4 + \dots$$

estimate using $P_2(x)$

$$e^{-2x} \approx 1 - 2x + \frac{(-2)^2}{2!}x^2 = 1 - 2x + 2x^2$$

with remainder

$$R_2(x) = \frac{(-2)^3}{3!}x^3 + \frac{(-2)^4}{4!}x^4 + \dots = \frac{f'''(c)}{3!}x^3$$

$\overset{0}{a} < c < x$

$$f'''(x) = -8e^{-2x}$$

$$\frac{f'''(x)}{3!}x^3 = \frac{-8e^{-2x}}{3!}x^3$$

$$\frac{f'''(c)}{3!}x^3 = -\frac{4}{3}\overset{?}{e^{-2c}}x^3 \qquad 0 < c < x$$

now let's use $P_2(x)$ to estimate $e^{-0.02}$ (as an example)

$$e^{-2x} \approx P_2(x) = 1 - 2x + 2x^2$$

$$e^{-0.02} \approx P_2(0.01) = 1 - 2(0.01) + 2(0.01)^2 = \frac{4901}{5000} \quad (0.9802)$$

what is the error?

it is exactly equal to $|R_2(c)|$ but we don't know c

$$\underset{a}{0} < c < \underset{x}{0.01}$$

can we at least bound e^{-2c} ?

e^{-2x} is a monotonic decreasing function, so on $0 < c < 0.01$

the maximum of $|e^{-2c}|$ is at the left end $(x = 0)$

$|e^{-2c}| \le |e^{-2(0)}| \le 1$ the max $|e^{-2c}|$ can be is 1

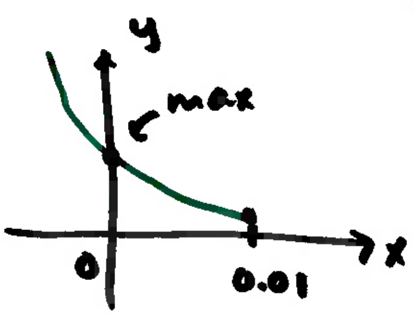
$f'''(c) =$

so, $|R_2(x)| \le \left| \cancel{\frac{1}{3!} (0.01)^3} \right|$

$\le \left| -\frac{4}{3} \underbrace{\cancel{f'''(c)}\, e^{-2c}}_{1} (0.01)^3 \right| = \frac{4}{3000000}$

so, our approx. of $e^{-0.02} = \frac{4901}{5000}$ is no more than $\frac{4}{3000000}$ off

$$\frac{4901}{5000} - \frac{4}{3000000} \le e^{-0.02} \le \frac{4901}{5000} + \frac{4}{3000000}$$



$$e^{-2x} \approx 1 - 2x + \frac{(-2)^2}{2!}x^2 + \frac{(-2)^3}{3!}x^3 + \frac{(-2)^4}{4!}x^4 + \frac{(-2)^5}{5!}x^5 + \dots$$

as example, to estimate, for example, e^{-1} using P_3

$$e^{-2x} \approx 1 - 2x + \frac{(-2)^2}{2!}x^2 + \frac{(-2)^3}{3!}x^3$$

$$e^{-1} = e^{-2\left(\underbrace{\frac{1}{2}}_{x}\right)} \approx 1 - 2\left(\frac{1}{2}\right) + \frac{(-2)^2}{2!}\left(\frac{1}{2}\right)^2 + \frac{(-2)^3}{3!}\left(\frac{1}{2}\right)^3$$

$$\approx 1 - 1 + \left(\frac{1}{2}\right) - \frac{4}{3}\left(\frac{1}{8}\right)$$

$$\approx \frac{1}{2} - \frac{1}{6} \approx \cancel{\frac{1}{12}}\ \frac{1}{3}$$

$$|\text{error}| = R_3(x) = \frac{f^{(4)}(a)}{4!}x^4 + \frac{f^{(5)}(a)}{5!}x^5 + \dots$$

$$= \left|\frac{f^{(4)}(c)}{4!}x^4\right|$$

bound $|f^{(4)}(c)|$ instead of finding c