11.2 Properties of Power Series (part 1)

power series : $\sum_{k=0}^{\infty} c_k (x-a)^k$ a: center c_k : coefficients of k^{th}-order term

convergence in general depends on x

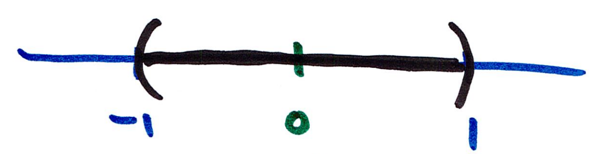
the interval on which the power series converges → interval of convergence
(already saw this in Ratio Test section)

for example, $\sum_{k=0}^{\infty} x^k$ $c_k = 1$, $a = 0$

converges if $|x| < 1$

$-1 < x < 1$

Sketch:



this is the radius of convergence (center to one end)

<u>example</u>

$$\sum_{k=1}^{\infty} \underbrace{\frac{(-1)^k k}{4^k}}_{c_k} \underbrace{(x+3)^k}_{(x-a)^k}$$

so $a = -3$ (center of interval of convergence)

find interval of convergence

Ratio Test is good for this

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{\frac{(-1)^{k+1}(k+1)}{4^{k+1}} (x+3)^{k+1}}{\frac{(-1)^k (k)}{4^k} (x+3)^k} \right| = \lim_{k\to\infty} \left| \frac{(-1)^{k+1}(k+1)}{4^{k+1}} \cdot \frac{4^k}{(-1)^k (k)} (x+3) \right|$$

$$= \lim_{k\to\infty} \left| (-1) \underbrace{\frac{(k+1)}{k}}_{1} \cdot \frac{1}{4} \cdot (x+3) \right| = \left| \frac{x+3}{4} \right| < 1$$

ratio < 1 converges

$$|x+3| < 4$$

$$-4 < x+3 < 4$$

$$-7 < x < 1$$

this is NOT it! At ends the ratio is 1 (inconclusive)

$$\sum_{k=1}^{\infty} \frac{(-1)^k k}{4^k} (x+3)^k$$

at $x=-7$ $\quad \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{4^k} (4^k) = \cancel{\sum_{k=1}^{\infty} (-1)^k} k$ $\quad$ diverges at $x=-7$

$$= \sum_{k=1}^{\infty} k$$

at $x=1$ $\quad \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{4^k} (4^k) = \sum_{k=1}^{\infty} (-1)^k k$ $\quad$ diverges at $x=1$

so interval of convergence is $-7 < x < 1$ or $(-7, 1)$

radius of conv.
$R = 4$

<u>example</u> $\displaystyle\sum_{k=1}^{\infty} (kx)^k = \sum_{k=1}^{\infty} k^k \cdot x^k = \sum_{k=1}^{\infty} \underbrace{k^k}_{c_k} \cdot (x-0)^k$ $\leftarrow a=0$

Ratio Test

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(k+1)^{k+1} \cdot x^{k+1}}{k^k \cdot x^k} \right| < 1$$

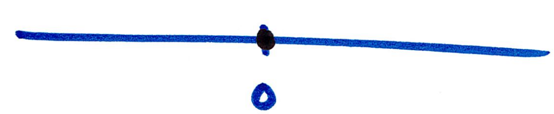
$$= \lim_{k\to\infty} \left| \underbrace{\left(\frac{k+1}{k}\right)^k}_{\substack{(1+\frac{1}{k})^k \\ e}} \cdot \underbrace{(k+1)}_{\infty} \cdot x \right| = \infty \quad \text{unless } x=0$$

ratio < 1 only if $x=0$

the only value of x for which the series converges is $x=0$

interval of convergence : $x=0$

radius of conv: $R=0$



series when $x=0$ $\displaystyle\sum_{k=1}^{\infty} k^k \cdot (0)^k = \sum_{k=1}^{\infty} 0 = 0+0+0+0+0+0+\cdots$

Summation notation is useful when finding intervals of convergence

you may need to find it yourself

<u>example</u> $x - \frac{x^3}{4} + \frac{x^5}{9} - \frac{x^7}{16} + \cdots$ in summation notation?

patterns? numerator : x to powers of odd numbers, alternating

denominator : something squared

$$= \underset{k=1}{\frac{x^1}{1^2}} - \underset{k=2}{\frac{x^3}{2^2}} + \underset{k=3}{\frac{x^5}{3^2}} - \underset{k=4}{\frac{x^7}{4^2}} + \underset{k=5}{\frac{x^9}{5^2}} - \underset{k=6}{\frac{x^{11}}{6^2}} + \cdots$$

decide a starting k

here, $k=1$

$$= \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{2k-1}}{k^2}$$

power series as functions

we know $\sum_{k=0}^{\infty} ar^k = a + ar^1 + ar^2 + ar^3 + \cdots = \frac{a}{1-r}$ if $|r| < 1$

treat it as a function : $r = x$, $a = 1$

$$\frac{a}{1-r} = \boxed{\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = \underbrace{1 + x + x^2 + x^3 + x^4 + \cdots}_{\text{Taylor series of } \frac{1}{1-x}} \quad \text{if } |x| < 1}$$

(without taking derivatives!)

we can modify it to find Taylor series of similar functions

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{k=0}^{\infty} (-x)^k = \sum_{k=0}^{\infty} (-1)^k x^k$$

now replace x in $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ w/ $(-x)$

$$= 1 - x + x^2 - x^3 + x^4 - x^5 + \cdots$$

converges if $|-x| < 1$

$|x| < 1$

$\frac{x^2}{1+x^2}$ Taylor series?

$$= x^2 \cdot \underbrace{\frac{1}{1-(-x^2)}}_{\text{reuse } \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \text{, change } x \text{ to } -x^2} = x^2 \sum_{k=0}^{\infty} (-x^2)^k = x^2 \sum_{k=0}^{\infty} (-1)^k x^{2k} = \sum_{k=0}^{\infty} (-1)^k x^{2k+2}$$

$|-x^2| < 1$

$x^2 < 1$

$-1 < x < 1$

$$= x^2 (1 - x^2 + x^4 - x^6 + x^8 - \cdots)$$

$$= x^2 - x^4 + x^6 - x^8 + x^{10} - \cdots.$$

the "1" in $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ is important

$\frac{1}{2-x}$ must rearrange so the denom. starts w/ 1

$$= \frac{1}{2(1-\frac{x}{2})} = \frac{1}{2} \cdot \frac{1}{\underbrace{1-(\frac{x}{2})}_{\text{reuse}}} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^k = \frac{1}{2} \sum_{k=0}^{\infty} \frac{x^k}{2^k} = \sum_{k=0}^{\infty} \frac{x^k}{2^{k+1}}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$$

$$|x| < 1$$

interval of conv: we changed x to $\frac{x}{2}$

$$|\text{"}x\text{"}| < 1 \implies \left|\frac{x}{2}\right| < 1$$

$$|x| < 2$$

$$-2 < x < 2$$