11.4 Working with Taylor Series

Taylor series: $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

any differentiable $f(x)$ near $x=a$ can be expressed as a Taylor series

if $a=0$, Taylor $\rightarrow$ Maclaurin

usually we work with "model" Maclaurin series

Common Maclaurin Series   a = 0 only

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k, \quad \text{for } |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^k x^k + \cdots = \sum_{k=0}^{\infty} (-1)^k x^k, \quad \text{for } |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad \text{for } |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^k x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + \frac{(-1)^{k+1} x^k}{k} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^k}{k}, \quad \text{for } -1 < x \le 1$$

$$-\ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^k}{k} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k}, \quad \text{for } -1 \le x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + \frac{(-1)^k x^{2k+1}}{2k+1} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}, \quad \text{for } |x| \le 1$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{x^{2k+1}}{(2k+1)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \quad \text{for } |x| < \infty$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^{2k}}{(2k)!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \quad \text{for } |x| < \infty$$

__example__

$$f(x) = \frac{9\tan^{-1}x - 9x + 3x^3}{5x^5}$$ Maclaurin series ?

build from Table of "model" series

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots$$

$$9\tan^{-1}x = 9x - 3x^3 + \frac{9}{5}x^5 - \frac{9}{7}x^7 + \frac{9}{9}x^9 - \cdots$$

$$9\tan^{-1}x - 9x = -\frac{9}{3}x^3 + \frac{9}{5}x^5 - \frac{9}{7}x^7 + \frac{9}{9}x^9 - \cdots$$

$$9\tan^{-1}x - 9x + 3x^3 = \frac{9}{5}x^5 - \frac{9}{7}x^7 + \frac{9}{9}x^9 - \frac{9}{11}x^{11} + \cdots$$

$$\frac{9\tan^{-1}x - 9x + 3x^3}{5x^5} = \frac{9}{5\cdot5} - \frac{9}{5\cdot7}x^2 + \frac{9}{5\cdot9}x^4 - \frac{9}{5\cdot11}x^6 + \frac{9}{5\cdot13}x^8 - \cdots$$

$$= \frac{9}{5}\Big(\underbrace{\frac{1}{5}}_{k=0} - \underbrace{\frac{x^2}{7}}_{k=1} + \underbrace{\frac{x^4}{9}}_{k=2} - \underbrace{\frac{x^6}{11}}_{k=3} + \frac{x^8}{13} - \frac{x^{10}}{15} + \cdots\Big)$$ choose to start at $k=0$

$$= \boxed{\frac{9}{5}\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{2k+5}}$$ near $x=0$, this behaves like $f(x)$

how to use that?

$$\lim_{x\to 0} \frac{9\tan^{-1}x - 9x + 3x^3}{5x^5}$$

usually we use l'Hospital's Rule

implies x is near 0
so Maclaurin series
can be used in place
of $f(x)$

$$= \lim_{x\to 0} \underbrace{\frac{9}{5}\left(\frac{1}{5} - \frac{x^2}{7} + \frac{x^4}{9} - \cdots\right)}_{\text{Maclaurin of } f(x)} = \frac{9}{25}$$

another good use

→ implies x is near 0

$$\int_0^1 \frac{9\tan^{-1}x - 9x + 3x^3}{5x^5}\,dx$$

$$= \int_0^1 \frac{9}{5}\left(\frac{1}{5} - \frac{x^2}{7} + \frac{x^4}{9} - \frac{x^6}{11} + \cdots\right)dx$$

$$= \frac{9}{5}\left(\frac{1}{5}x - \frac{x^3}{3\cdot 7} + \frac{x^5}{5\cdot 9} - \frac{x^7}{7\cdot 11} + \frac{x^9}{9\cdot 13} - \frac{x^{11}}{11\cdot 15} + \cdots\right)\Bigg|_0^1$$

$$= \underbrace{\frac{9}{5}\left(\frac{1}{5} - \frac{1}{21} + \frac{1}{45}\right.}_{\text{approx. of the integral}} - \frac{1}{77} + \frac{1}{117} - \cdots \Big)$$

let's cut off after $\cancel{\frac{1}{44}}$ $\frac{1}{45}$

$$\int_0^1 \frac{9\tan^{-1}x - 9x + 3x^3}{5x^5}\,dx \approx \frac{9}{5}\left(\frac{1}{5} - \frac{1}{21} + \frac{1}{45}\right) \approx 0.3143$$

what is the |error|?

one option: Taylor's Remainder Theorem

easier

Alternating Series Estimation (because this happens to be an alt. series)

$$|\text{error}| \le |\text{first term we throw out}|$$

$$\le \left|\frac{9}{5}\cdot -\frac{1}{77}\right| \approx 0.0234$$

example Estimate $\int_0^{0.1} e^{-x^2}\,dx$ to within 10^{-8}

near 0 → Maclaurin is good

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \cdots$$

$$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots$$

$$\int_0^{0.1} e^{-x^2}\,dx = \int_0^{0.1}\left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots\right)dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \frac{x^9}{9\cdot 4!} - \frac{x^{11}}{11\cdot 5!} + \frac{x^{13}}{13\cdot 6!} - \cdots \Big|_0^{0.1}$$

$$= 0.1 - \underbrace{\frac{(0.1)^3}{3}}_{3\times 10^{-4}} + \underbrace{\frac{(0.1)^5}{5\cdot 2!}}_{10^{-6}} - \underbrace{\frac{(0.1)^7}{7\cdot 3!}}_{2.38\times 10^{-9}} + \frac{(0.1)^9}{9\cdot 4!} - \cdots$$

alt. series

$|\text{error}| \leq |$ first term we throw out $|$

goal: $|\text{error}| \leq 10^{-8}$

$\downarrow$

first time below 10^{-8}

so sum up to the term before it

$$\approx 0.1 - \frac{(0.1)^3}{3} + \frac{(0.1)^5}{5\cdot 2!} \approx 0.099068 \approx \int_0^{0.1} e^{-x^2}\,dx \quad (\text{within } 10^{-8})$$

$$\int e^{-x}\,dx = -e^{-x} + C$$

$$\int e^{-x^2}\,dx = \text{cannot be expressed using elementary functions in closed form}$$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$ Alt. Harmonic converges

$= S$ (sum of this series)

has a sum because it converges

$S = ?$

from Table : $\ln(1+x) = x - \underbrace{\frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5}}_{\text{alt. harmonic if } x = 1} - \cdots$

$\ln(1+1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \boxed{\ln(2)}$

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