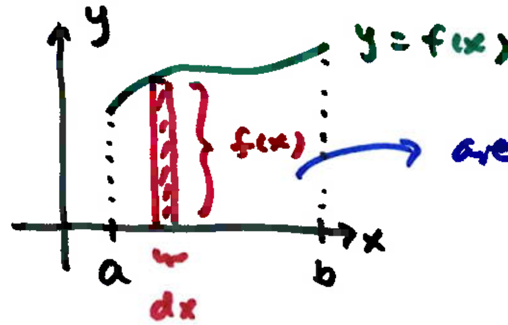
6.2 Regions Between Curves

The area under $y=f(x)$ above ~~y~~ x-axis between $x=a$ and $x=b$

is $\int_a^b \underbrace{f(x)\,dx}$

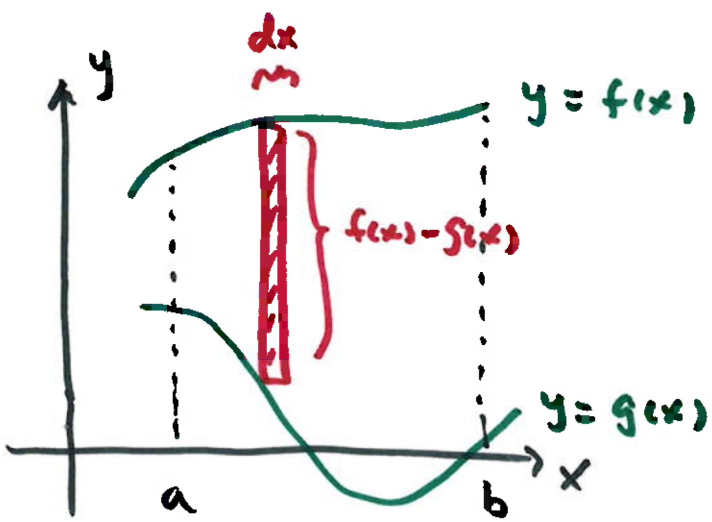
area of a thin rectangle
height $f(x)$
width dx



area $= \int_a^b f(x)\,dx$

Sum of infinitely many $f(x)\,dx$ as x goes from a to b

this idea can be extended to find area between two curves



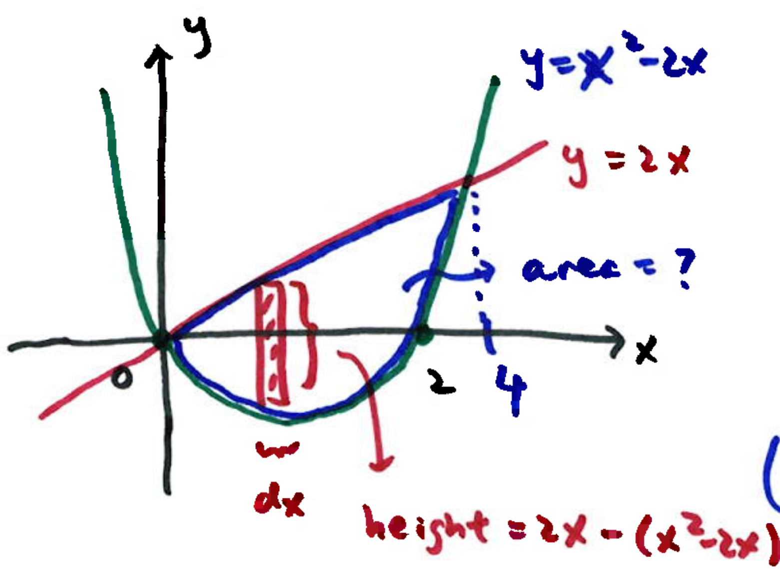
each rectangle has area $= [f(x) - g(x)]\,dx$

use integration to accumulate

$$\boxed{\int_a^b [\underbrace{f(x)}_{\text{top}} - \underbrace{g(x)}_{\text{bottom}}]\,dx}$$

example Find the area of region bounded by

$$\underbrace{y = x^2 - 2x}_{\text{parabola opens up}} \quad \text{and} \quad \underbrace{y = 2x}_{\text{line}}$$



$y = x^2 - 2x$

$= x(x-2) \rightarrow$ x-intercepts $(y=0)$ at $x = 0$, $x = 2$

the left end of the region :
the right end of the region :

intersections of $y = x^2 - 2x$ and $y = 2x$

$x^2 - 2x = 2x$

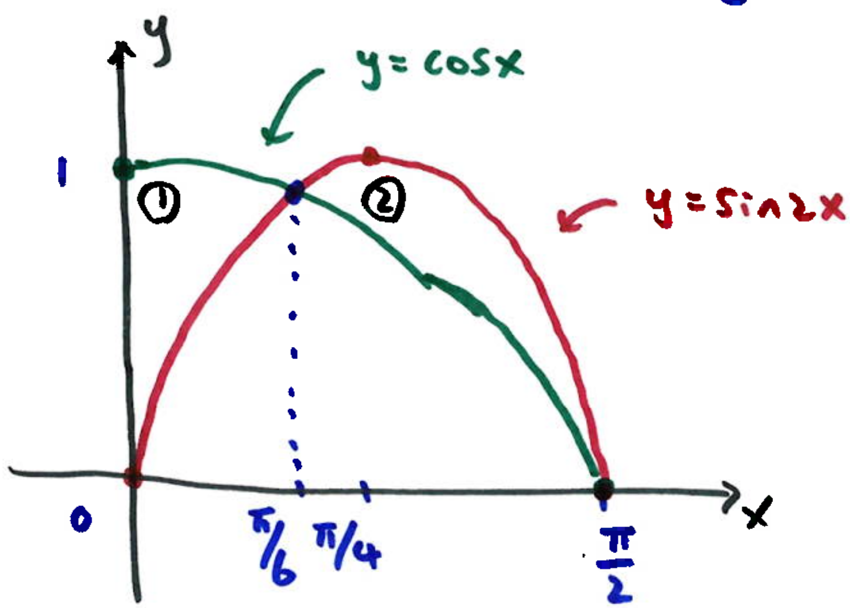
$x^2 - 4x = 0 \rightarrow x(x-4) = 0 \quad x = 0, x = 4$

rectangle height : $2x - (x^2 - 2x) = 4x - x^2$

" width : dx

area of region: $\displaystyle\int_0^4 (4x - x^2)\,dx = 4 \cdot \frac{x^2}{2} - \frac{x^3}{3}\Big|_0^4 = 2x^2 - \frac{1}{3}x^3\Big|_0^4 = \left(32 - \frac{64}{3}\right) - 0 = \boxed{\frac{32}{3}}$

example Find area of region bound by $y=\cos x$ and $y=\sin 2x$ between $x=0$ and $x=\frac{\pi}{2}$



$\sin\frac{\pi}{2} = 1$

$\sin 2x = 1 \rightarrow 2x = \frac{\pi}{2}$

$x = \frac{\pi}{4}$

1: $\cos x$ above, $\sin 2x$ below

2: $\sin 2x$ above, $\cos x$ below

intersection: $\cos x = \sin 2x$

identity: $\sin 2x = 2\sin x\cos x$

$\cos x = 2\sin x\cos x$

$\cos x - 2\sin x\cos x = 0$

$\cos x(1-2\sin x) = 0$

$\cos x = 0$, $1-2\sin x = 0$

$\downarrow$ $\qquad$ $\sin x = \frac{1}{2}$

$x = \pi/2$ $\qquad$ $x = \frac{\pi}{6}$

$$\underbrace{\int_0^{\frac{\pi}{6}} (\cos x - \sin 2x)\,dx}_{①} + \underbrace{\int_{\pi/6}^{\pi/2} (\sin 2x - \cos x)\,dx}_{②}$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sin x\,dx = -\cos x + C$$

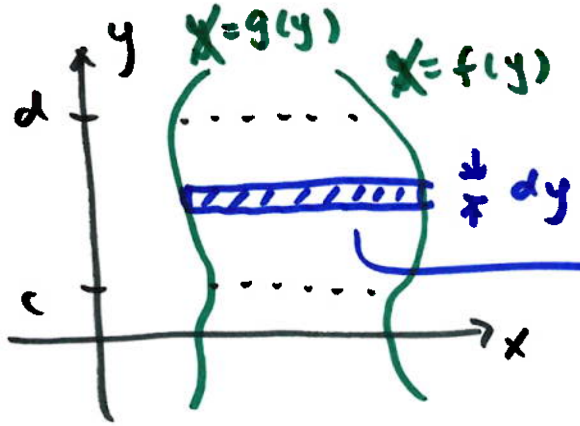
$$= \left(\sin x + \frac{1}{2}\cos 2x\right)\Big|_0^{\pi/6} + \left(-\frac{1}{2}\cos 2x - \sin x\right)\Big|_{\pi/6}^{\pi/2}$$

$$= \left(\sin\frac{\pi}{6} + \frac{1}{2}\cos\frac{\pi}{3}\right) - \left(\sin 0 + \frac{1}{2}\cos 0\right)$$

$$+ \left(-\frac{1}{2}\cos\pi - \sin\frac{\pi}{2}\right) - \left(-\frac{1}{2}\cos\frac{\pi}{3} - \sin\frac{\pi}{6}\right)$$

$$= \left(\frac{1}{2} + \frac{1}{4}\right) - \left(0 + \frac{1}{2}\right) + \left(\frac{1}{2} - 1\right) - \left(-\frac{1}{4} - \frac{1}{2}\right) = \boxed{\frac{1}{2}}$$

if we can integrate in terms of x, we can integrate in terms of y



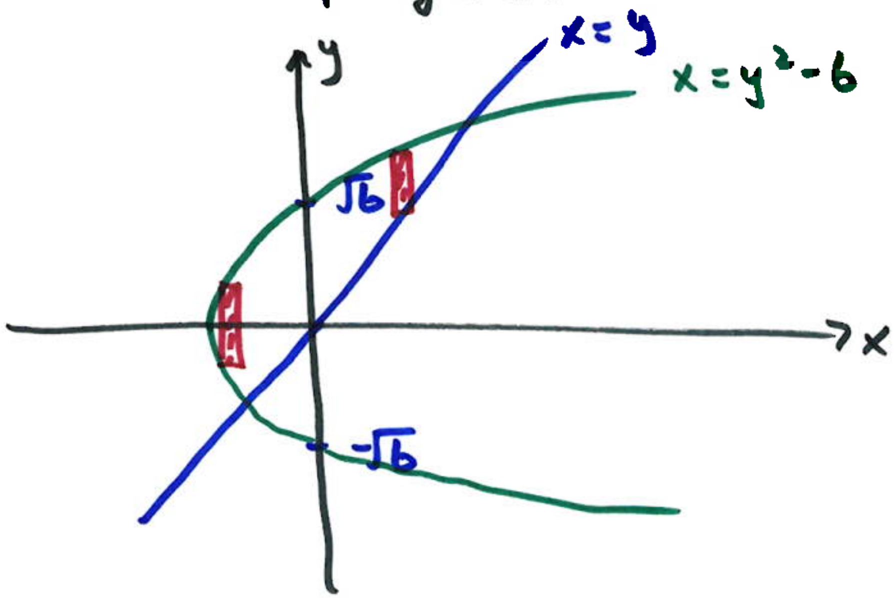
width : $f(y) - g(y)$
right left

area $= \int_c^d [f(y) - g(y)]\,dy$

<u>example</u> $x = y^2 - 6$, $x = y$

parabola opening RIGHT

↳ line

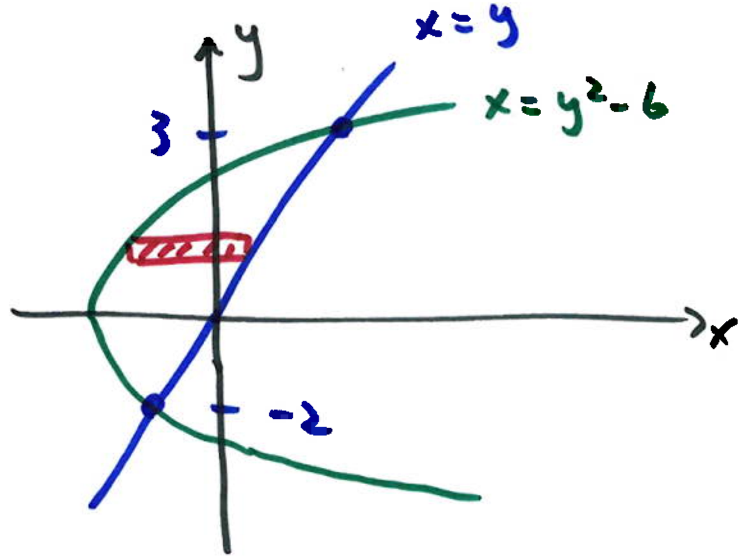


$x = y^2 - 6$ find y-ints ($x=0$)

$0 = y^2 - 6$ $y = \pm\sqrt{6}$

if we use vertical rectangles, then the roles of "top" and "bottom" switch/change at some point

→ ok, but maybe there is something easier horizontal rectangles are a little easier



this time, "right" is always $x = y$

"left" is always $x = y^2 - 6$

no switching!

intersections: $y^2 - 6 = y$

$y^2 - y - 6 = 0$

$(y-3)(y+2) = 0$

$y = -2, \ y = 3$

$$\int_{-2}^{3} [\underbrace{y}_{\text{right}} - \underbrace{(y^2-6)}_{\text{left}}]\,dy = \int_{-2}^{3} (y - y^2 + 6)\,dy$$

$$= \frac{y^2}{2} - \frac{y^3}{3} + 6y \Big|_{-2}^{3} = \cdots = \boxed{\frac{125}{6}}$$