example $\vec{F} = \langle x^2 - z e^y,\ y^3 - xz e^y,\ z^4 - x e^y \rangle$

$\qquad\qquad f \qquad\qquad g \qquad\qquad h$

is $f_y = g_x$? $\quad -ze^y = -ze^y$ yes

$g_z = h_y$? $\quad -xe^y = -xe^y$ yes

$f_z = h_x$? $\quad -e^y = -e^y$ yes

ALL 3 must be yes for $\vec{F} = \nabla \phi$

now we know

$\vec{F} = \langle f, g, h \rangle = \langle \phi_x, \phi_y, \phi_z \rangle$

$\phi_x = x^2 - ze^y$ ①

$\phi_y = y^3 - xze^y$ ②

$\phi_z = z^4 - xe^y$ ③

from ①: $\phi = \int (x^2 - ze^y)\,dx = \frac{1}{3}x^3 - xze^y + a(y,z)$ ④

y, z are constants

$a(y,z)$: function that can depend on y, z

what is a?

partial of ④ with y must be ②

$\phi_y = -xze^y + \frac{\partial a}{\partial y} = \underbrace{y^3 - xze^y}_{②} \rightarrow \frac{\partial a}{\partial y} = y^3$ ⑤

partial of ④ with z must be ③

$\phi_z = -xe^y + \frac{\partial a}{\partial z} = \underbrace{z^4 - xe^y}_{③} \rightarrow \frac{\partial a}{\partial z} = z^4$ ⑥

integrate ⑤ with y : $a = \frac{1}{4}y^4 + b(z)$

take partial with z and compare to ⑥

$\frac{\partial a}{\partial z} = \frac{db}{dz} = z^4 \rightarrow b = \frac{1}{5}z^5 + C$ so $a = \frac{1}{4}y^4 + \frac{1}{5}z^5 + C$