Find an equation of the plane that contains the point $(2,1,1)$ and the line

$$x = 1 + 3t, \quad y = 2 + t, \quad z = 4 + t.$$

A. $3x + y + z = 8$

B. $2x + y + z = 6$

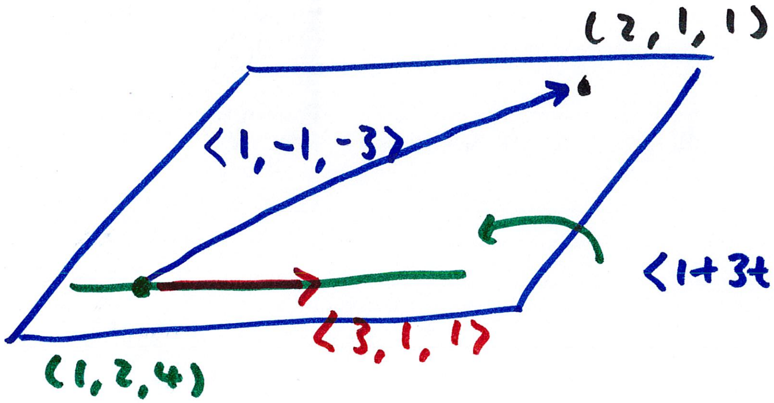
C. $x + 2y + 4z = 8$

D. $x - 5y + 2z = -1$

E. $x - 2y + z = 1$

plane : $\underbrace{\hat{n}}_{\text{normal vector}} \cdot \underbrace{\langle x - x_0, y - y_0, z - z_0 \rangle}_{\text{vector in plane}} = 0$

(x_0, y_0, z_0) in plane



$$\langle 1+3t, 2+t, 4+t \rangle = \underbrace{\langle 1, 2, 4 \rangle}_{\text{one point on the line}} + t \underbrace{\langle 3, 1, 1 \rangle}_{\text{direction vector}}$$

vectors in plane : $\vec{u} = \langle 1, -1, -3 \rangle$

$\vec{v} = \langle 3, 1, 1 \rangle$

$\vec{n} = \vec{u} \times \vec{v}$ or $\vec{v} \times \vec{u}$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & -3 \\ 3 & 1 & 1 \end{vmatrix} = \vec{i}(2) - \vec{j}(10) + \vec{k}(4)$$

$$= \langle 2, -10, 4 \rangle$$

plane: $\langle 2, -10, 4 \rangle \cdot \langle x-2, y-1, z-1 \rangle = 0$

$$2(x-2) - 10(y-1) + 4(z-1) = 0$$

$$2x - 4 - 10y + 10 + 4z - 4 = 0$$

$$2x - 10y + 4z = -2$$

An object is traveling on the trajectory given by $r(t) =< 3t^2 - 1, 4t^2 + 5 >, t \geq 0$. What is the object's speed at the moment when it has traveled a distance of 5 on this trajectory?

distance or length : $s(t) = \int_a^t |\vec{r}\,'(u)|\, du$

speed: $|\vec{r}\,'(t)|$

$\vec{r}\,'(t) = \langle 6t, 8t \rangle$

$|\vec{r}\,'(t)| = \sqrt{36t^2 + 64t^2} = 10t$

find t that corresponds to "having traveled distance of 5"

$s(t) = \int_0^t 10u\, du = 5u^2 \Big|_0^t = 5t^2$

distance $= 5 \rightarrow 5t^2 = 5 \rightarrow t = 1$

speed at that time: $10(1) = 10$

A. $\frac{1}{10}$

B. $\frac{1}{5}$

C. 5

D. 10

E. 50

The limit $\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{x^2+y^2}$ is equal to

A. 1

B. 0

C. 1/2

D. 2

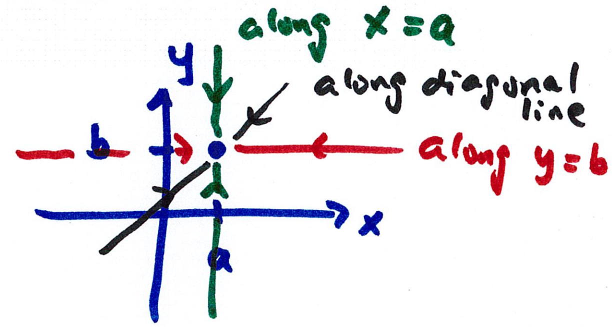
E. Does not exist

limit: show that limit is path dependent

(in which case limit DNE)

or

find limit by path independent way



Simply paths to check:

along $x=0$: $\lim_{y\to 0} \frac{y^4}{y^2} = \lim_{y\to 0} y^2 = 0$

along $y=0$: $\lim_{x\to 0} \frac{x^4}{x^2} = \lim_{x\to 0} x^2 = 0$

Still not enough to prove the limit is 0

along $y=x$: $\lim_{x\to 0} \frac{x^4+x^4}{x^2+x^2} = \lim_{x\to 0} \frac{2x^4}{2x^2} = \lim_{x\to 0} x^2 = 0$

time to think of a path independent way

back to $\lim_{(x,y)\to(0,0)} \dfrac{x^4+y^4}{x^2+y^2}$

if numerator were x^4-y^4

then it's $(x^2-y^2)(x^2+y^2)$

but it's not

but the numerator looks a lot like $(x^2+y^2)^2$

$(x^2+y^2)^2 = x^4+2x^2y^2+y^4$

so $x^4+y^4 = (x^2+y^2)^2-2x^2y^2$

$$\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)^2-2x^2y^2}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)^2}{x^2+y^2} - \frac{2x^2y^2}{x^2+y^2}$$

to 0

$\to$ (small #)^4 when $(x,y)\to(0,0)$

$\to$ (small #)^2

(small #)^4 goes to 0 faster so the 2nd part also $\to 0$

$= 0$

If $f(x,y) = \ln(xy^2 + x)$ find f_{xy}.

A. 0

B. $\frac{-y^2-1}{(xy^2+x)^2}$

C. $\frac{-2y}{(xy^2+x)^2}$

D. $\frac{-y^2}{(xy^2+x)^2}$

E. $\frac{-2xy}{(xy^2+x)^2}$

$f(x,y) = \ln(xy^2 + x)$ find f_{xy}

y is const

$$f_x = \frac{1}{xy^2+x} \frac{\partial}{\partial x}(xy^2+x)$$

$$= \frac{1}{xy^2+x}(y^2+1)$$

$$f_x = \frac{y^2+1}{xy^2+x}$$

x const

$$f_{xy} = \frac{(xy^2+x)\frac{\partial}{\partial y}(y^2+1) - (y^2+1)\frac{\partial}{\partial y}(xy^2+x)}{(xy^2+x)^2}$$

(quotient rule)

$$= \frac{(xy^2+x)(2y) - (y^2+1)(2xy)}{(xy^2+x)^2} = \frac{2xy^3 + 2xy - 2xy^3 - 2xy}{(xy^2+x)^2}$$

If $f(x, y) = x^2 + 3y^2$, for which unit vector $\vec{u}$ does the directional derivative $(D_{\vec{u}}f)(2, 1)$ have a minimum value?

unit vector giving direction

directional deriv : $\vec{\nabla} f \cdot \vec{u}$

gradient

A. $4\vec{i} + 6\vec{j}$

B. $-4\vec{i} - 6\vec{j}$

C. $\frac{2}{\sqrt{13}}\vec{i} + \frac{3}{\sqrt{13}}\vec{j}$

D. $\frac{-2}{\sqrt{13}}\vec{i} - \frac{3}{\sqrt{13}}\vec{j}$

E. $\frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{j}$

gradient is in direction of greatest increase in $f(x,y)$
(greatest ascent or max directional deriv)

min directional deriv : opposite to direction of gradient

0 directional deriv → so $\perp \vec{\nabla} f$ (along a level curve)

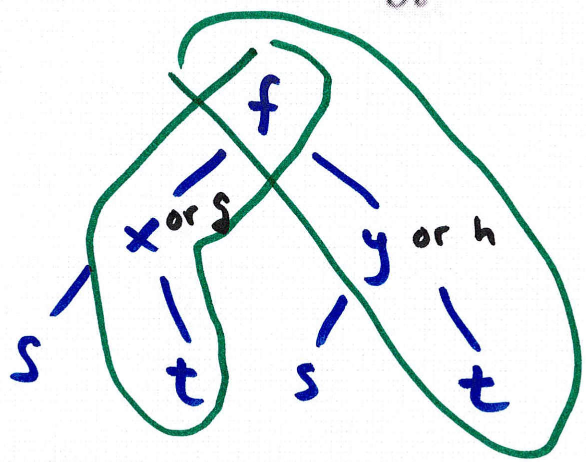
$\vec{\nabla} f = \langle f_x, f_y \rangle = \langle 2x, 6y \rangle$

$\vec{\nabla} f(2,1) = \langle 4, 6 \rangle$ make unit vector : $\frac{\langle 4, 6 \rangle}{|\langle 4, 6 \rangle|} = \frac{\langle 4, 6 \rangle}{\sqrt{52}}$

$\left\langle \frac{4}{\sqrt{52}}, \frac{6}{\sqrt{52}} \right\rangle = \left\langle \frac{4}{2\sqrt{13}}, \frac{6}{2\sqrt{13}} \right\rangle$

$= \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$ so opposite

Let $f(x,y) = 2x^2y + xy^3$ and $x = g(s,t)$, $y = h(s,t)$ are functions of s and t. Suppose $g(1,2) = 1$, $h(1,2) = -1$ and $\frac{\partial g}{\partial t}(1,2) = 2$, $\frac{\partial h}{\partial t}(1,2) = 1$. Then at $(s,t) = (1,2)$, $\boxed{\frac{\partial f}{\partial t}}$ equals



A. 0

B. -10

C. 10

D. 5

E. -5

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

evaluate at $s=1$, $t=2$ $\rightarrow$ $x(1,2) = g(1,2) = 1$ $\leftarrow$ given

$y(1,2) = h(1,2) = -1$

$$\frac{\partial f}{\partial x} = 4xy + y^3 = -4 - 1 = -5$$

$$\frac{\partial f}{\partial y} = 2x^2 + 3xy^2 = 2 + 3 = 5$$

$$\frac{\partial f}{\partial t} = (-5)(2) + (5)(1) = -5$$

other topics to review: surface identification (need to know their names)
(and level curves)

implicit differentiation: chain rule

curvature : $\kappa = \dfrac{|\vec{T}'(t)|}{|\vec{r}'|} = \dfrac{|\vec{r}'' \times \vec{r}'|}{|\vec{r}'|^3}$