Let $\mathbf{F} = 4x\mathbf{i} - z\mathbf{j} + x\mathbf{k}$. Compute $\overbrace{\iint_S \mathbf{F} \cdot d\mathbf{S}}^{\text{Flux}}$, where S is the union of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, and the base given by $x^2 + y^2 \leq 1$, $z = 0$. (Use the outward–pointing normal.)

A. $\frac{2\pi}{3}$

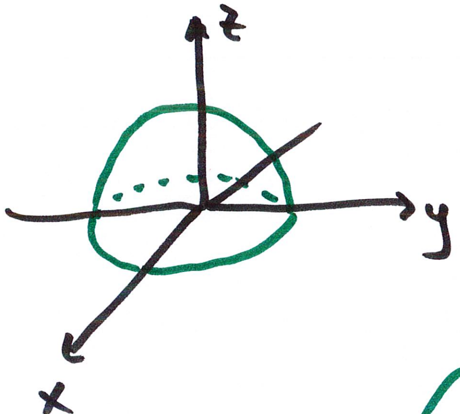
B. $\frac{16\pi}{3}$

C. $\frac{8\pi}{3}$

D. $\frac{4\pi}{3}$

E. $\frac{-4\pi}{3}$

Surface closed: can use Divergence Theorem
if $\vec{F}$ is defined everywhere
inside the enclosed volume



Div. Theorem:

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

$$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \langle 4x, -z, x \rangle = 4 + 0 + 0 = 4$$

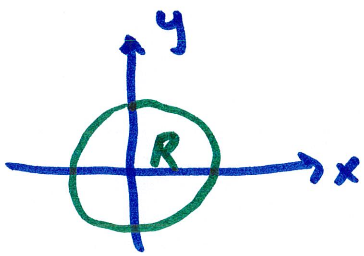
$$\iiint_E 4 \, dV = 4 \underbrace{\iiint_E dV}_{\text{volume of hemisphere}} = 4 \cdot \frac{1}{2} \cdot \frac{4}{3} \pi (1)^3 = \frac{16}{6} \pi = \frac{8\pi}{3}$$

Flux

note if $\operatorname{div} \vec{F} = 1$ then $\iint_S \vec{F} \cdot d\vec{S} = \text{volume}$

very much like we can use Green's Theorem to find area

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (g_x - f_y)\, dA$$



if $\vec{F}$ is such that $g_x - f_y = 1$

then $\oint_C \vec{F}\cdot d\vec{r} =$ area of R

Find $\overbrace{\iint_S \vec{F}\cdot\vec{n}\,dS}^{\text{Flux}}$ for the region bounded between $\overbrace{x^2+z^2=4}^{\text{cylinder}}$, $(-2\le y\le 2)$ and $\overbrace{x^2+y^2+z^2=1}^{\text{sphere}}$ given $\vec{F} = <3x+y, 2y+z, z>$

including $y=2$ and $y=-2$

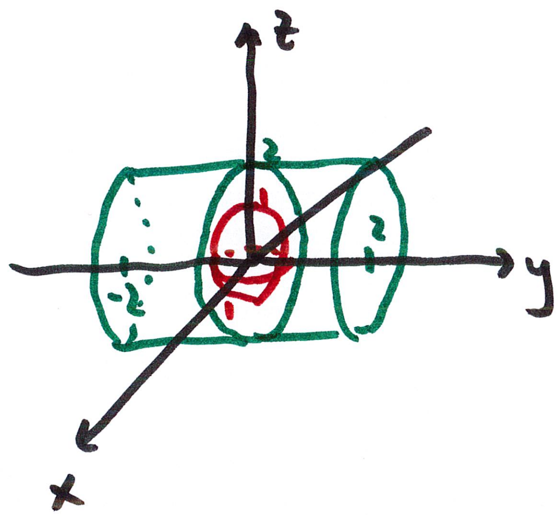
A. 0

B. $\frac{44\pi}{3}$

C. 88π

D. $\frac{188\pi}{3}$

E. 376π



flux through the surface bounding the space between sphere and capped cylinder

Divergence Theorem for hollowed volume

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_{\text{outside}} \operatorname{div}\vec{F}\,dV - \iiint_{\text{inside}} \operatorname{div}\vec{F}\,dV$$

$$\operatorname{div}\vec{F} = 3+2+1 = 6$$

$$\iiint_{cyl} 6\,dV - \iiint_{sph} 6\,dV = 6\left(\overbrace{\iiint_{cyl} dV}^{\text{vol. of cylinder}} - \iiint_{sph} dV\right)$$

$$= 6\left(\pi(2)^2\cdot 4 - \frac{4}{3}\pi\right) = 6\pi\left(16-\frac{4}{3}\right) = 6\pi\left(\frac{44}{3}\right)$$

The position of a particle is given by $\mathbf{r}(t) = \langle 2t, 1-2t, 5+t \rangle$, starting when $t = 0$. After the particle has gone a *distance* of 3, the x-coordinate is

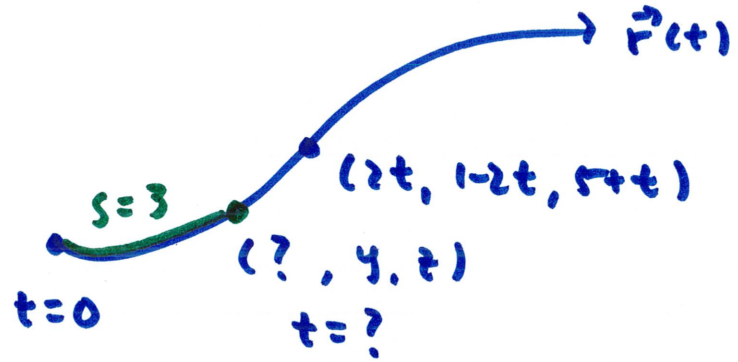
A. $\frac{1}{3}$

B. 3

C. $\frac{1}{2}$

D. 2

E. 1



length: $\quad s(t) = \int_0^t |\vec{r}\,'|\,du \qquad \vec{r}\,' = \langle 2, -2, 1 \rangle$

$|\vec{r}\,'| = \sqrt{4+4+1} = 3$

$= \int_0^t 3\,du = 3t$

want $s=3 \rightarrow 3 = 3t \rightarrow t=1$

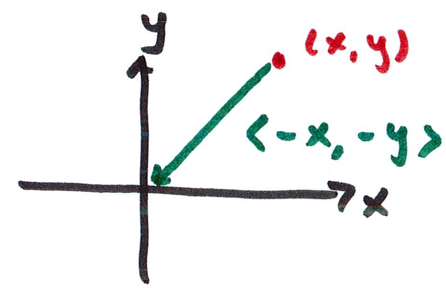
at $t=1$, $\vec{r}(1) = \langle 2, -1, 6 \rangle$

17 from practice problems set

Find directional derivative of $f(x,y) = 5-4x^2-3y$ at (x,y) towards the origin

$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$ ← unit vector specifying direction

↖ gradient



$$\vec{u} = \frac{\langle -x,-y\rangle}{\sqrt{x^2+y^2}}$$

$\vec{\nabla} f = \langle -8x, -3\rangle$

$$D_{\vec{u}} f = \langle -8x, -3\rangle \cdot \frac{\langle -x,-y\rangle}{\sqrt{x^2+y^2}} = \frac{8x^2+3y}{\sqrt{x^2+y^2}}$$

$\vec{\nabla} f$: direction? Greatest <u>increase</u> → $\perp$ to level curve

magnitude? maximum directional derivative

no $D_{\vec{u}} f$ if going $\perp$ $\vec{\nabla} f$ or parallel to level curve

#24 The function $f(x,y) = 2x^3 - 6xy - 3y^2$

find max/min/saddle pt

find critical pts: $f_x = 0$ AND $f_y = 0$

$f_x = 6x^2 - 6y = 0 \rightarrow x^2 = y \rightarrow x = y, -y$

$f_y = -6x - 6y = 0 \rightarrow \underbrace{x = -y}$

sub into $f_x = 0$

$6(-y)^2 - 6y = 0$

$6y^2 - 6y = 0$

$6y(y-1) = 0 \rightarrow y = 0, \; y = 1$

$x = 0, \; x = -1$

cp: $(0,0), (-1, 1)$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$f_{xx} = 12x \qquad f_{xy} = -6$

$f_{yy} = -6$

$D(x,y) = -72x - 36$

at $(0, 0)$

$D = -36 < 0 \rightarrow$ saddle pt

at $(-1, 1)$

$D = 72 - 36 > 0 \rightarrow$ max or min

check f_{xx} $f_{xx}(-1, 1) < 0 \rightarrow$ max

if $D = 0 \rightarrow$ inconclusive, this test fails

14.

$$f(x,y) = \cos(xy)$$

$$\underbrace{\frac{\partial^2 f}{\partial x \partial y}}_{\text{y first, then x}} = ?$$

"constant"

$$\frac{\partial}{\partial y}\left(\cos(xy)\right) = -\sin(xy)\cdot x = -x\sin(xy)$$

"constant"

$$\frac{\partial}{\partial x}\left[(-x)\cdot \sin(xy)\right]$$

$$= (-x)\cdot \frac{\partial}{\partial x}\sin(xy) + \sin(xy)\cdot \frac{\partial}{\partial x}(-x)$$

$$= (-x)\cdot \cos(xy)\cdot y + \sin(xy)\cdot (-1)$$

A potential for the vector field $\mathbf{F}(x, y) = \langle 3x^2y + 2xy,\ x^3 + x^2 \rangle$ is

A. $xy^3 + xy^2 + C$

B. $x^3y + x^2y + C$

C. $x^2y^3 + xy^3 + C$

D. $x^2y^3 + x^3 + C$

E. A potential for $\mathbf{F}$ does not exist.

$\vec{\nabla}\phi = \vec{F}$

$\langle \phi_x, \phi_y \rangle = \langle 3x^2y + 2xy,\ x^3 + x^2 \rangle$

$\phi_x = 3x^2y + 2xy \xrightarrow{\text{integrate}} \phi = \int (3x^2y + 2xy)\,dx = \underbrace{x^3y + x^2y + h(y)}_{\text{what we know about } \phi}$

$\phi_y = x^3 + x^2$

partial of our ϕ with respect to y: $\phi_y = x^3 + x^2 + \frac{dh}{dy}$

it must be equal to $\phi_y = x^3 + x^2$ (from $\vec{F}$)

so, $\frac{dh}{dy} = 0$ which means $h = C$ (numerical constant)

so, ~~$\phi = x^3 + x^2 + C$~~

$\phi = x^3y + x^2y + C$

ϕ is important in Fundamental Theorem of Line Integrals

$$\int_C \underbrace{\vec{\nabla}\phi}_{\vec{F}} \cdot d\vec{r} = \phi(\text{end}) - \phi(\text{start})$$: path independent

if $\vec{F}$ is conservative

$\hookrightarrow$ curl $\vec{F} = \vec{0}$