Let $\vec{F}(x,y,z) = 2y\vec{\imath} - x\vec{\jmath} + 3\vec{k}$. Which of the following statements is true?

~~(i)~~ Curl $\vec{F} = \vec{0}$.

~~(ii)~~ $\vec{F}$ is a gradient of some function f.

~~(iii)~~ Line integrals, $\int_C \vec{F} \cdot d\vec{r}$, over a curve C from a point P to a point Q, are path independent.

~~(iv)~~ $\int_C \vec{F} \cdot d\vec{r} = 0$ for all smooth closed curves C.

A. (i) only

B. (ii) only

C. (ii) and (iii) only

D. (ii), (iii) and (iv) only

E. None of the above

$\vec{F} = \langle 2y, -x, 3 \rangle$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & -x & 3 \end{vmatrix}$$

$$= \langle 0-0,\ 0-0,\ -1-2 \rangle = \langle 0, 0, -3 \rangle$$

if $\vec{F}$ is conservative, then $\vec{F} = \vec{\nabla} f$, and curl $\vec{F} = \vec{0}$

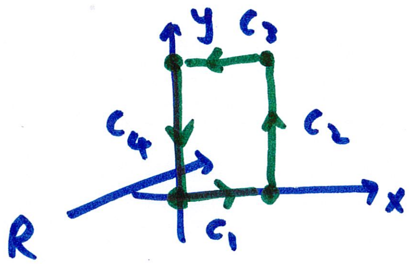
here, $\vec{F}$ is NOT conservative, so is NOT $\vec{\nabla} f$

iii) $\int_C \vec{F}\cdot d\vec{r}$ is path independent only if $\vec{F} = \vec{\nabla}\phi$

iv) no, for the same reason as iii)

Evaluate $\int_C y\,dx + \cos y\,dy$ where C is the polygonal path from $(0,0)$ to $(1,0)$ to $(1,2)$ to $(0,2)$ to $(0,0)$.

$\int_C \vec{F} \cdot d\vec{r}$



$\vec{F} = \langle y, \cos y \rangle = \langle f, g \rangle$

$d\vec{r} = \langle dx, dy \rangle$

A. -1

B. 0

C. -2

D. $-\frac{3}{2}$

E. $\frac{2}{3}$

two ways: parametrize paths C_1, C_2, C_3, C_4

then do $\int_C \vec{F} \cdot d\vec{r}$

Green's Theorem (path is closed)

$\oint_C \vec{F} \cdot d\vec{r} = \iint_R (g_x - f_y)\, dA$ here, $g_x - f_y = 0 - 1 = -1$

$\iint_R (-1)\, dA = (-1) \underbrace{\iint_R dA}_{\text{area of } R} = (-1)(1)(2) = -2$

or $= (-1) \int_0^1 \int_0^2 dy\,dx$

Identify the surface $2x^2 + 3z^2 = 4x + 2y^2$ through completing the square.

~~A.~~ Cone

~~B.~~ Ellipsoid (signs)

C. Parabolic hyperboloid

D. Hyperboloid of one sheet

E. Hyperboloid of two sheets

$$2x^2 - 4x - 2y^2 + 3z^2 = 0$$

$$2(x^2 - 2x) - 2y^2 + 3z^2 = 0$$

$$2(x^2 - 2x + 1) - 2y^2 + 3z^2 = 2$$

$$2(x-1)^2 - 2y^2 + 3z^2 = 2$$

$$(x-1)^2 - y^2 + \frac{3}{2}z^2 = 1$$

center : $(1, 0, 0)$

let $x - 1 = w$

$$w^2 - y^2 + \frac{3}{2}z^2 = 1$$

wy-trace $(z=0)$: $w^2 - y^2 = 1$ hyp

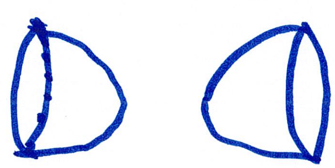
w-ints: ± 1

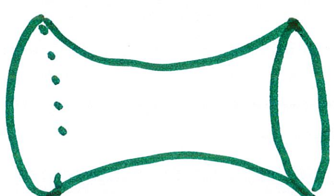
wz-trace $(y=0)$: $w^2 + \frac{3}{2}z^2 = 1$ ellipse

yz-trace $(w=0)$: $\frac{3}{2}z^2 - y^2 = 1$ hyp

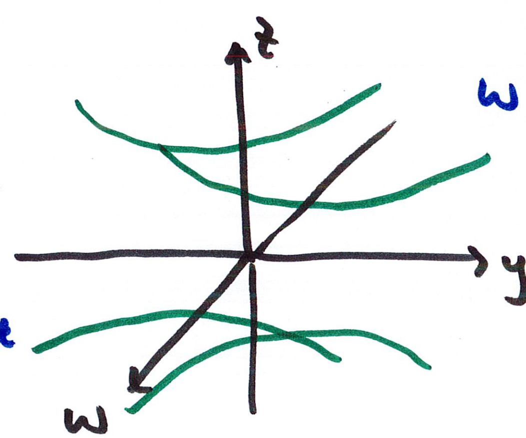
Para. hyp: saddle / pringle shape

Two-sheets:





hyp of ONE-sheet

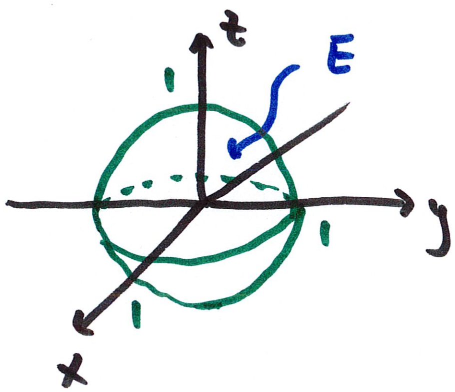


If $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + z\vec{k}$, Σ is the unit sphere $x^2+y^2+z^2=1$ and $\vec{n}$ is the outward unit normal on Σ, then $\iint_\Sigma \vec{F}\cdot\vec{n}\,dS =$

A. -4π B. $\frac{2\pi}{3}$ C. 0 D. $\frac{4\pi}{3}$ E. 4π

$\vec{F} = \langle x, y, z\rangle$ $\Sigma : x^2+y^2+z^2=1$ $\vec{n}$: outward

calculate flux $\iint_\Sigma \vec{F}\cdot\vec{n}\,dS$



choices: parametrize Σ, find $\vec{r_u}\times\vec{r_v}$ or $\vec{r_v}\times\vec{r_u}$ (whichever is out)

then do $\iint_R \vec{F}\cdot(\vec{r_u}\times\vec{r_v})\,dA$

$\rightarrow$ Divergence Theorem: closed surface

$\vec{F}$ defined in the volume

Div. Theorem: $\iint_S \vec{F}\cdot\vec{n}\,dS = \iiint_E \operatorname{div}\vec{F}\,dV$

$\operatorname{div}\vec{F} = 3$

$$\iiint_E \operatorname{div}\vec{F}\,dV = 3\overbrace{\iiint_E dV}^{\text{volume of } E \text{ (sphere)}} = 3\cdot\frac{4}{3}\pi(1)^3 = 4\pi$$

1. Find an equation of the plane that contains the point $(1, 2, -3)$ and the line with symmetric equations $x - 2 = y - 1 = \dfrac{z+2}{2}$.

A. $5x + y + z = 4$

B. $2x - y + z = -3$

C. $3x + y - 2z = 11$

D. $4x - 2y - 3z = 9$

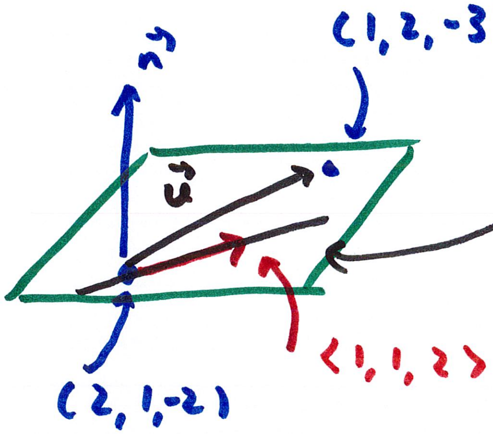
E. $x + y - 2z = 9$

Sym. eq of line : $t = x - 2 = y - 1 = \dfrac{z+2}{2}$

$x = 2 + t,$

$y = 1 + t$

$z = 2t - 2$



$\vec{r}(t) = \langle 2+t,\ 1+t,\ 2t-2 \rangle$

$= \langle 2, 1, -2 \rangle + t \underbrace{\langle 1, 1, 2 \rangle}_{\vec{v}}$

$\vec{u} = \langle -1, 1, -1 \rangle$

$\vec{n} = \vec{u} \times \vec{v}$ or $\vec{v} \times \vec{u}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix} = \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \langle 3, 1, -2 \rangle \end{array}$$

plane: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$3(x - 1) + (y - 2) - 2(z + 3) = 0$

An object occupies the region bounded above by the sphere $\overbrace{x^2 + y^2 + z^2}^{\text{radius } \sqrt{32}} = 32$ and below by the upper nappe of the cone $z^2 = x^2 + y^2$. The mass density at any point of the object is equal to its distance from the xy plane. Set up a triple integral in rectangular coordinates for the total mass m of the object.

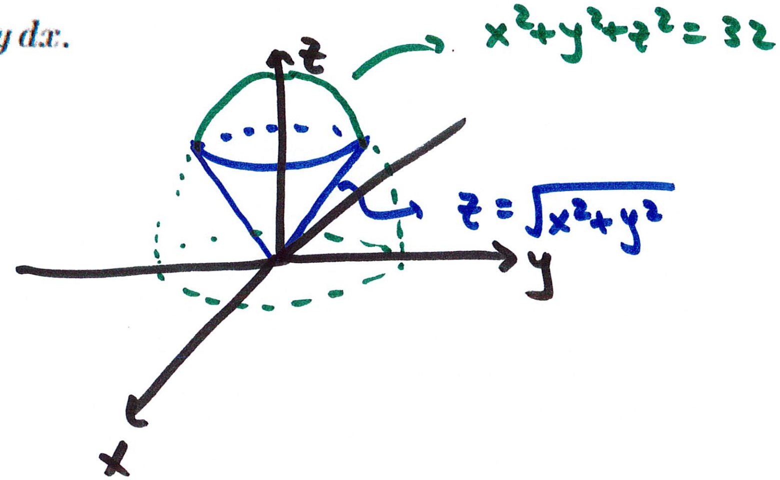
A. $\int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{-\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} z\, dz\, dy\, dx$

B. $\int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} z\, dz\, dy\, dx$

C. $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} z\, dz\, dy\, dx$

D. $\int_{0}^{4} \int_{0}^{\sqrt{16-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} z\, dz\, dy\, dx$

E. $\int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} xy\, dz\, dy\, dx$.



Now do the same in spherical

A. $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{32}} \rho^3 \cos\varphi \sin\varphi\, d\rho\, d\varphi\, d\theta$

B. $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{32}} \rho \cos\varphi \sin\varphi\, d\rho\, d\varphi\, d\theta$

C. $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{32}} \rho^3 \sin^2\varphi\, d\rho\, d\varphi\, d\theta$

D. $\int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{32}} \rho^3 \cos\varphi \sin\varphi\, d\rho\, d\varphi\, d\theta$

E. $\int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{32}} \rho \cos\varphi\, d\rho\, d\varphi\, d\theta$.

density : $\rho(x,y,z) = z$

in Cartesian : intersection of surfaces projected on xy-plane

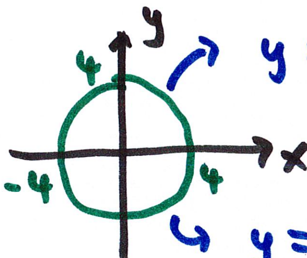
$x^2+y^2+z^2 = 32 \to z = \sqrt{32-x^2-y^2}$

$z = \sqrt{x^2+y^2}$

$\sqrt{x^2+y^2} = \sqrt{32-x^2-y^2}$

$x^2+y^2 = 32-x^2-y^2$

$x^2+y^2=16$



$y = \sqrt{16-x^2}$

$y = -\sqrt{16-x^2}$

$-4 \le x \le 4$

$-\sqrt{16-x^2} \le y \le \sqrt{16-x^2}$

$\sqrt{x^2+y^2} \le z \le \sqrt{32-x^2-y^2}$

(sphere)

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{32-x^2-y^2}} z \, dz\, dy\, dx$$

Spherical: $0 \le \rho \le \sqrt{32}$ $\quad 0 \le \theta \le 2\pi$

$0 \le \phi \le \pi/4$

density: $\rho = z = \rho\cos\phi$

$dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{32}} \rho^3 \cos\phi \sin\phi \, d\rho\, d\phi\, d\theta$$