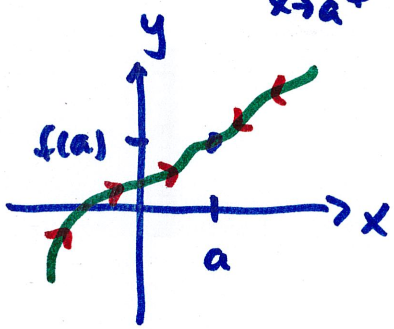
15.2 Limits and Continuity

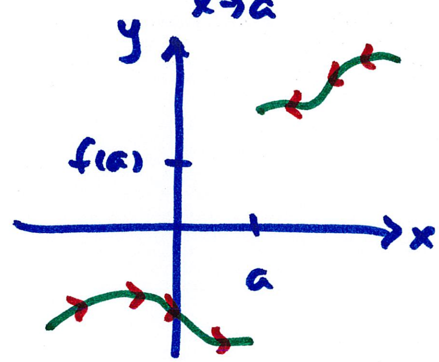
recall if $\lim_{x \to a} f(x) = L$ then we can make $f(x)$ as close to L as we want by making x sufficiently close to a.

if $\lim_{x \to a} f(x) = L$ then it doesn't matter how we approach a

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a} f(x) = L$$

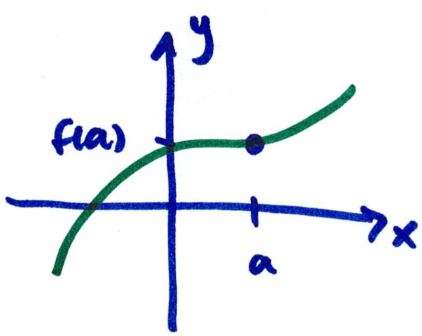


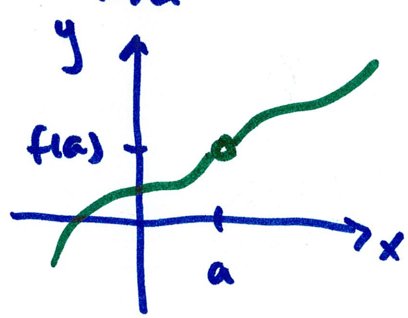
limit exists



limit DNE

if $f(x)$ is continuous at $x = a$, then $\lim_{x \to a} f(x) = f(a)$





limit exists
but $f(a)$ not defined
NOT continuous

many types of functions are continuous everywhere

polynomial, sine, cosine, exponential

many types are continuous wherever defined

rational, logarithmic

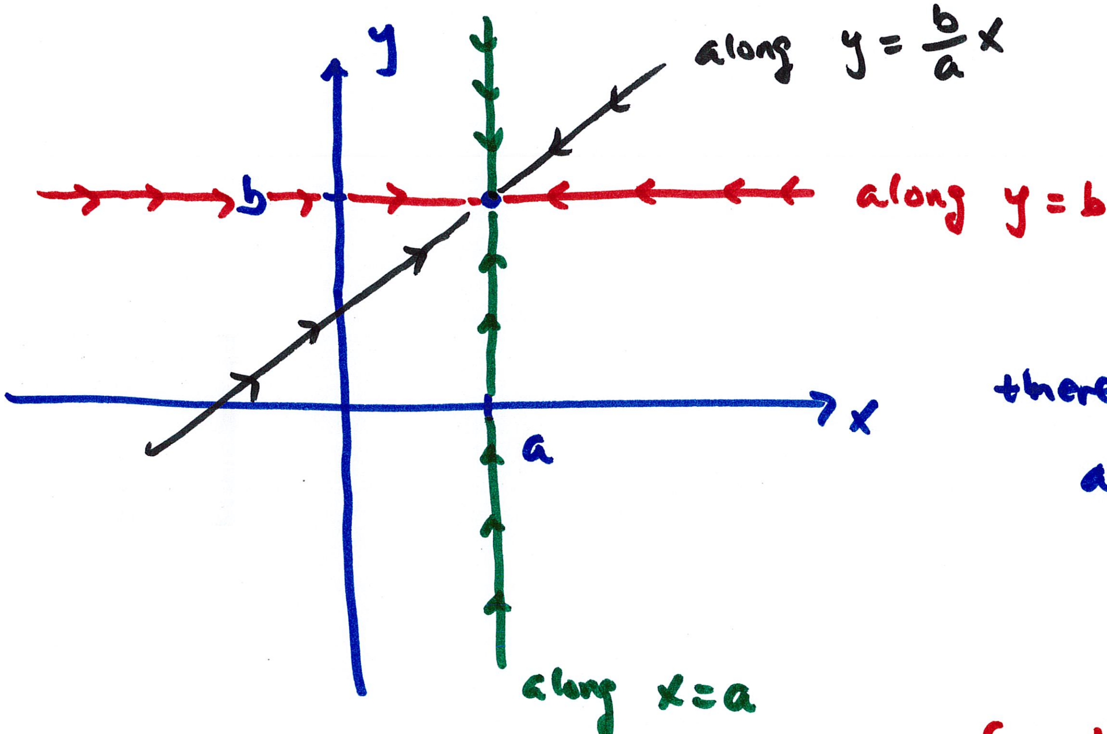
almost everything we just reviewed carry directly over to functions of two variables

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

we can make $f(x,y)$ as close to L as we want by making ~~(a,b~~ (x,y) sufficiently close to (a,b)

whether (x,y) can equal (a,b) is NOT relevant for limit just like before, if $f(x,y)$ has a limit at (a,b), it should NOT depend on how we get there.

but now we have more ways to get to (a,b)



there are many, many more ways! along a parabola, exponential, sine, etc

for limit to exist, it should NOT depend on path

→ how can we check infinitely-many paths?

<u>example</u> $$\lim_{(x,y)\to(1,0)} \ln\left(\frac{1+y^2}{x^2+xy}\right)$$

we don't have to worry about path because this is a logarithmic function and it is continuous if defined

$\rightarrow$ if continuous, $f(a,b) = \lim\limits_{(x,y)\to(a,b)} f(x,y)$

is $\ln\left(\frac{1+y^2}{x^2+xy}\right)$ defined at $(1,0)$?

$x \quad y$

$$\ln\left(\frac{1+0}{1+0}\right) = \ln(1) = 0$$ yes!

so this function is continuous at $(1,0)$

therefore, $f(1,0) = \boxed{\lim\limits_{(x,y)\to(1,0)} f(x,y) = 0}$

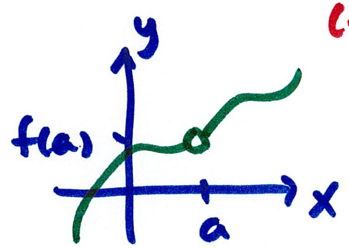
example $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2-4x^2}{2x^2+y^2}$

this is a rational function which is continuous wherever defined.

but here, it is not defined at (0,0) so is not continuous at (0,0)

so we can't use continuity to find limit

just because a function is not continuous at (a,b), it does NOT automatically mean $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y)$ DNE



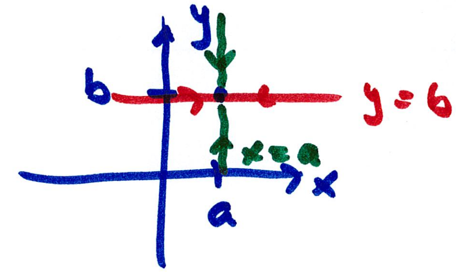
limit is still f(a)

so now we need to worry about path

→ all paths should lead to same limit for limit to exist

→ if two paths lead to different numbers, then limit DNE

$$\lim_{(\underset{a}{x},\underset{b}{y})\to(0,0)} \frac{y^2-4x^2}{2x^2+y^2}$$



the two easiest paths to try are $x=a$ and $y=b$

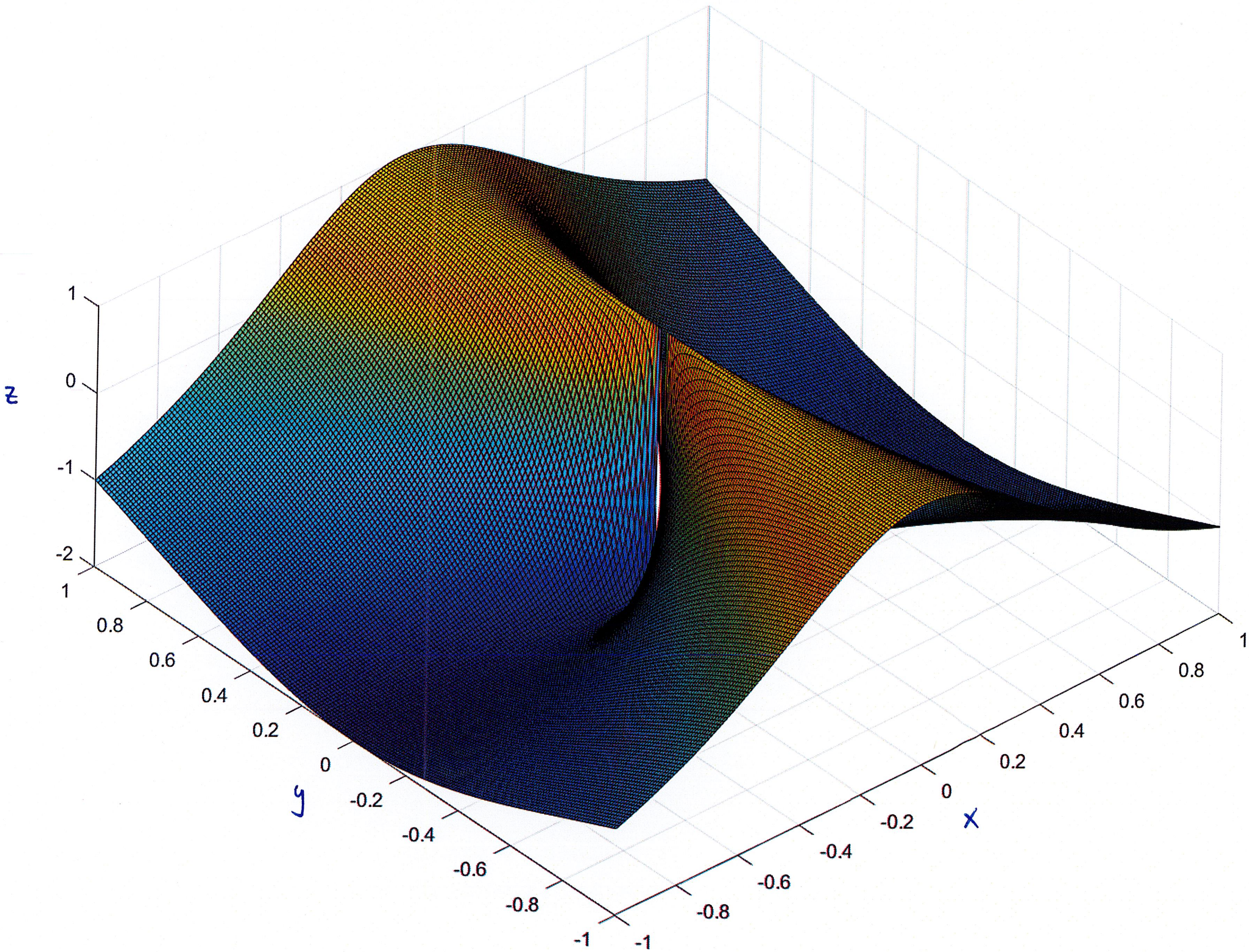
along $x=a=0$

$$\lim_{(x,y)\to(0,0)} \frac{y^2-4\overset{0}{x^2}}{2\underset{0}{x^2}+y^2} = \lim_{y\to 0} \frac{y^2}{y^2} = 1$$

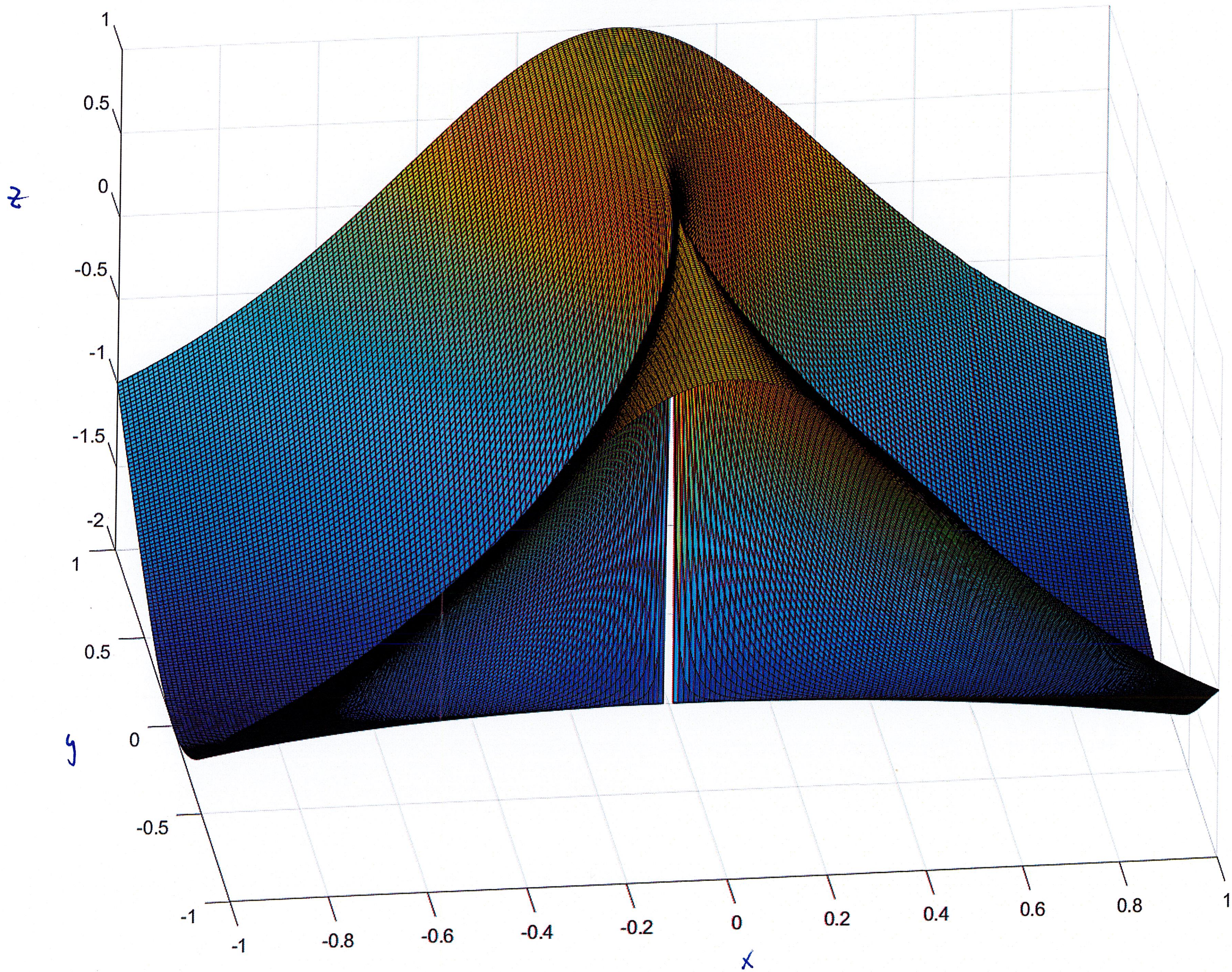
along $y=b=0$

$$\lim_{(x,y)\to(0,0)} \frac{\overset{0}{y^2}-4x^2}{2x^2+\underset{0}{y^2}} = \lim_{x\to 0} \frac{-4x^2}{2x^2} = -2$$

Since two paths lead to different numbers, the limit at $(0,0)$ DNE

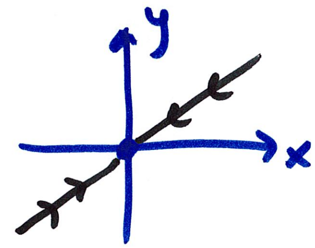


z
1
0
-1
-2
y
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
x
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1



1
0.5
0
-0.5
-1
-1.5
-2
z
1
0.5
0
-0.5
-1
y
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
x

<u>example</u> $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}$



clearly $\dfrac{x^2-y^2}{x+y}$ DNE at $(0,0)$

along $x=0$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{y\to 0} \frac{-y^2}{y} = 0$

along $y=0$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{x\to 0} \frac{x^2}{x} = 0$

this is NOT enough! we need to see ALL paths lead to same number

next path? $y = x$

along $y=x$: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\overset{y}{x}{}^2-y^2}{\underset{y}{x}+y} = \lim_{y\to 0} \frac{y^2-y^2}{y+y} = 0$

what's next? $y=x^2$? $y=x^3$? $y=\sin x$?

we can't possibly check all paths

so find a way to work with f(x,y) w/o assuming a path

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{\cancel{(x+y)}(x-y)}{\cancel{(x+y)}} = \lim_{(x,y)\to(0,0)} (x-y)$$

goes to 0 w/o assuming path

goes to 0 no matter the path

$$= \boxed{0}$$

without assuming a path

limit is 0

example

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$$

clearly, continuity doesn't help

along $x=0$, $y=0$ same thing but not enough to conclude

path independent way?

recall $$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

notice $(x,y) \to (0,0)$ $\quad x^2+y^2 \to 0$ no matter how we get there

let $u = x^2+y^2$

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{u\to 0} \frac{\sin(u)}{u} = 1$$