15.4 The Chain Rule

if $y = f(x)$ and $x = g(t)$ then the Chain Rule is used to find the rate of change of y with respect to t

$$\frac{dy}{dt} = \frac{dy}{dx}\,\frac{dx}{dt}$$

how t affects y

how y is affected by its own variable x

how t affects x

a good way to track variable relationships is a tree diagram

$$\begin{tikzcd} y \arrow[d, dash] \arrow[d, bend left, "\frac{dy}{dx}"] \arrow[dd, bend right=60, "\frac{dy}{dt}"'] \\ x \arrow[d, dash] \arrow[d, bend left, "\frac{dx}{dt}"] \\ t \end{tikzcd}$$

each step down is a derivative

partial derivative → isolates effect from one variable

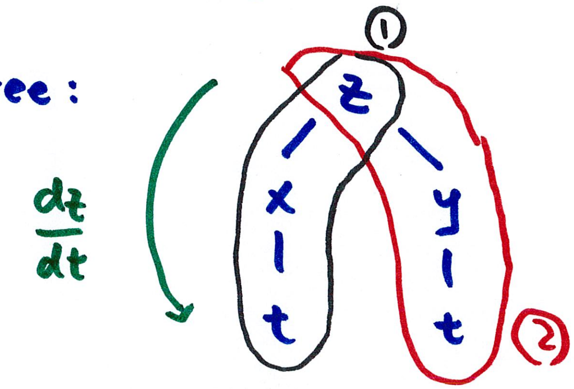
multiple variables : isolate effects, then combine

example $z = f(x,y) = x^3 + y^2$

$x = \cos(t) \quad y = \ln(t)$

find $\frac{dz}{dt}$

tree :



find how z is affected by t one branch at a time

① : pretend y is not doing anything

$$\frac{\partial z}{\partial x} \frac{dx}{dt}$$

partial down through a split

② : pretend x is doing nothing

$$\frac{\partial z}{\partial y} \frac{dy}{dt}$$

put them together

partial : split

regular : no split

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

put answer in terms of this

$$= (3x^2)(-\sin t) + (2y)\left(\frac{1}{t}\right)$$

$$= \boxed{-3\cos^2 t \sin t + 2\frac{\ln t}{t}}$$

Quick check: what if we sub in x, y as functions of t initially?

$z = x^3 + y^2 \qquad x = \cos t \qquad y = \ln t$

$z = (\cos t)^3 + (\ln t)^2$

$\frac{dz}{dt} = 3(\cos t)^2 \cdot -\sin t + 2(\ln t) \cdot \frac{1}{t}$ same

example

$$z = \sin(x+y)$$

$$x = u^2 + v$$

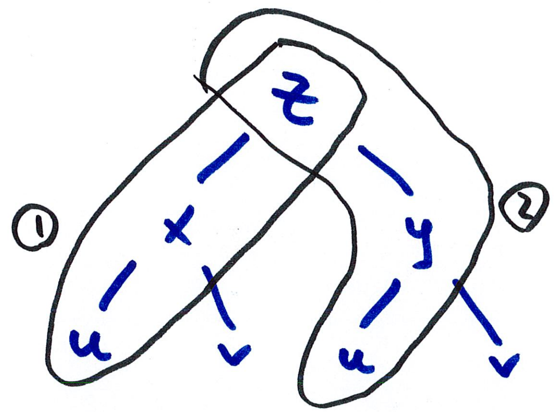
$$y = 1 - 2uv$$

ultimately z is a function of u and v

$\rightarrow z = f(u,v)$

find: $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$

tree :



use branches w/ u at bottom

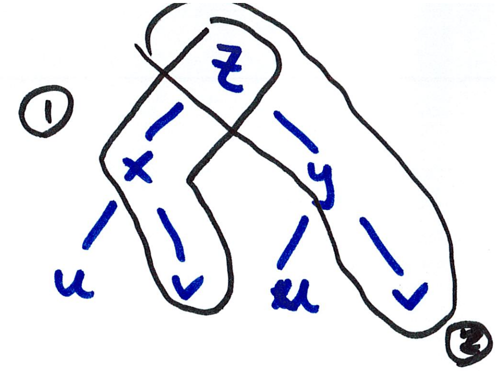
$$\frac{\partial z}{\partial u} = \underbrace{\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}}_{1} + \underbrace{\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}}_{2}$$

$$= \cos(x+y)\cdot 2u + \cos(x+y)(-2v)$$

put everything in terms of u (and possibly v)

$$= \cos(u^2+v+1-2uv)(2u) + \cos(u^2+v+1-2uv)(-2v)$$

$$\frac{\partial z}{\partial v} = \underbrace{\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}}_{①} + \underbrace{\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}}_{②}$$



$$= \cos(x+y)\cdot(1) + \cos(x+y)\cdot(-2u)$$

$$= \cos(x+y)\,(1-2u) = \boxed{\cos\left(u^2+v+1-2uv\right)(1-2u)}$$

implicit differentiation

$x^2+2y^2=4$ $\quad$ y is an implicit function of x $\quad$ ($y=f(x)$)

find $\frac{dy}{dx}$

from calc 1: $\quad \frac{d}{dx}\left(x^2+2y^2\right) = \frac{d}{dx}(4)$

$$2x + 4y\cdot\frac{dy}{dx} = 0$$

solve for $\frac{dy}{dx}$: $\quad \frac{dy}{dx} = -\frac{x}{2y}$

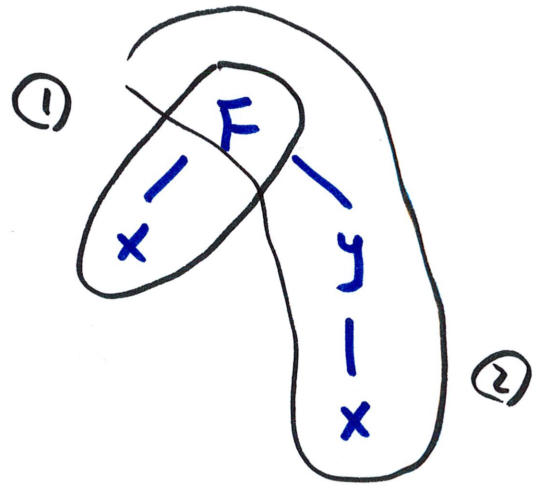
we can use the multivariable chain rule to find the same result

$x^2 + 2y^2 = 4$ $\quad$ y is an implicit function of x $\quad$ ($y = f(x)$)

create a new function: $F(x,y) = x^2 + 2y^2 - 4 = 0 = g(x)$

because y = some function of x

tree: $g(x) = F(x,y) = 0$



find $\dfrac{dy}{dx}$: step down the tree

we want this

because $g = 0$ so $\dfrac{dg}{dx} = 0$

$$\frac{dg}{dx} = \underset{①}{\frac{\partial F}{\partial x}} + \underset{②}{\frac{\partial F}{\partial y}} \boxed{\frac{dy}{dx}} = 0$$

solve for $\dfrac{dy}{dx}$: $\dfrac{dy}{dx} = \dfrac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\dfrac{F_x}{F_y} = -\dfrac{2x}{4y} = \boxed{-\dfrac{x}{2y}}$

can do this w/ more variables ("calc 1" way gets harder)

example

$$xy + yz + xz = 3$$

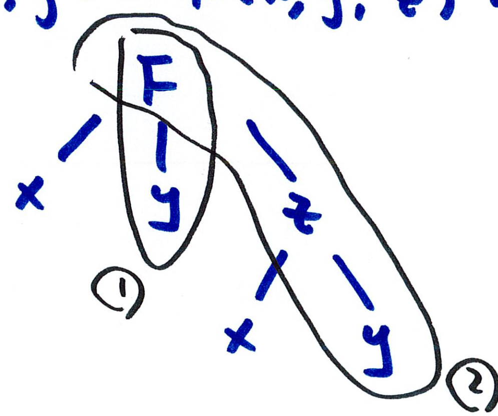
z is an implicit function of x and y $\longrightarrow$ $z = f(x,y)$

and x and y don't depend on anything else

find $\dfrac{\partial z}{\partial y}$

define $F(x,y,z) = xy + yz + xz - 3 = 0 = g(x,y)$

$g(x,y) = F(x,y,z) = 0$



$\dfrac{\partial z}{\partial y} =$: collect branches w/ y at base

$g(x,y) = F(x,y,z) = 0$

we want this

$$\frac{\partial}{\partial y}(g) = \underset{1}{\frac{\partial F}{\partial y}} + \underset{2}{\frac{\partial F}{\partial z}} \boxed{\frac{\partial z}{\partial y}} = 0$$

because g is constant

$$\frac{\partial z}{\partial y} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = \boxed{\frac{-(x+z)}{y+x}}$$