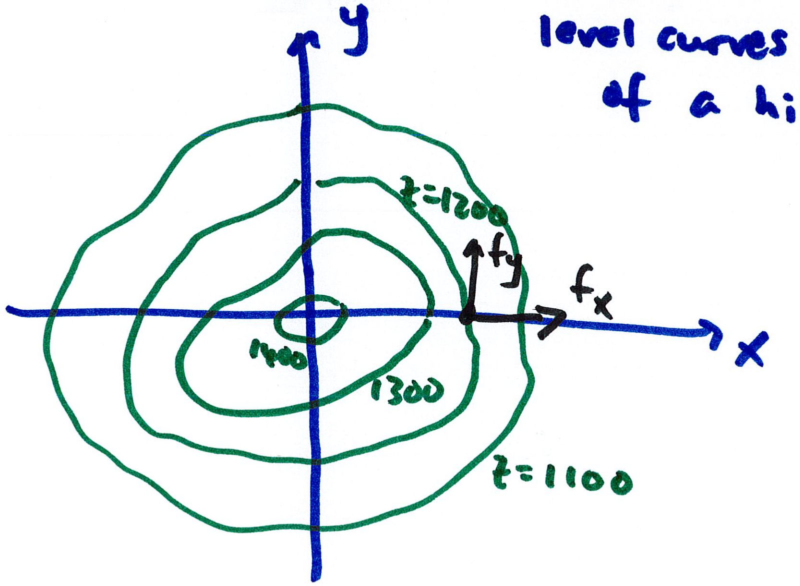
15.5 Directional Derivative and the Gradient

if $z = f(x, y)$

we know $\frac{\partial f}{\partial x} = f_x$ is the rate of change of f with respect to x while y is constant

$\frac{\partial f}{\partial y} = f_y$ same idea, but x is fixed and y varies.

there is a physical interpretation of f_x and f_y

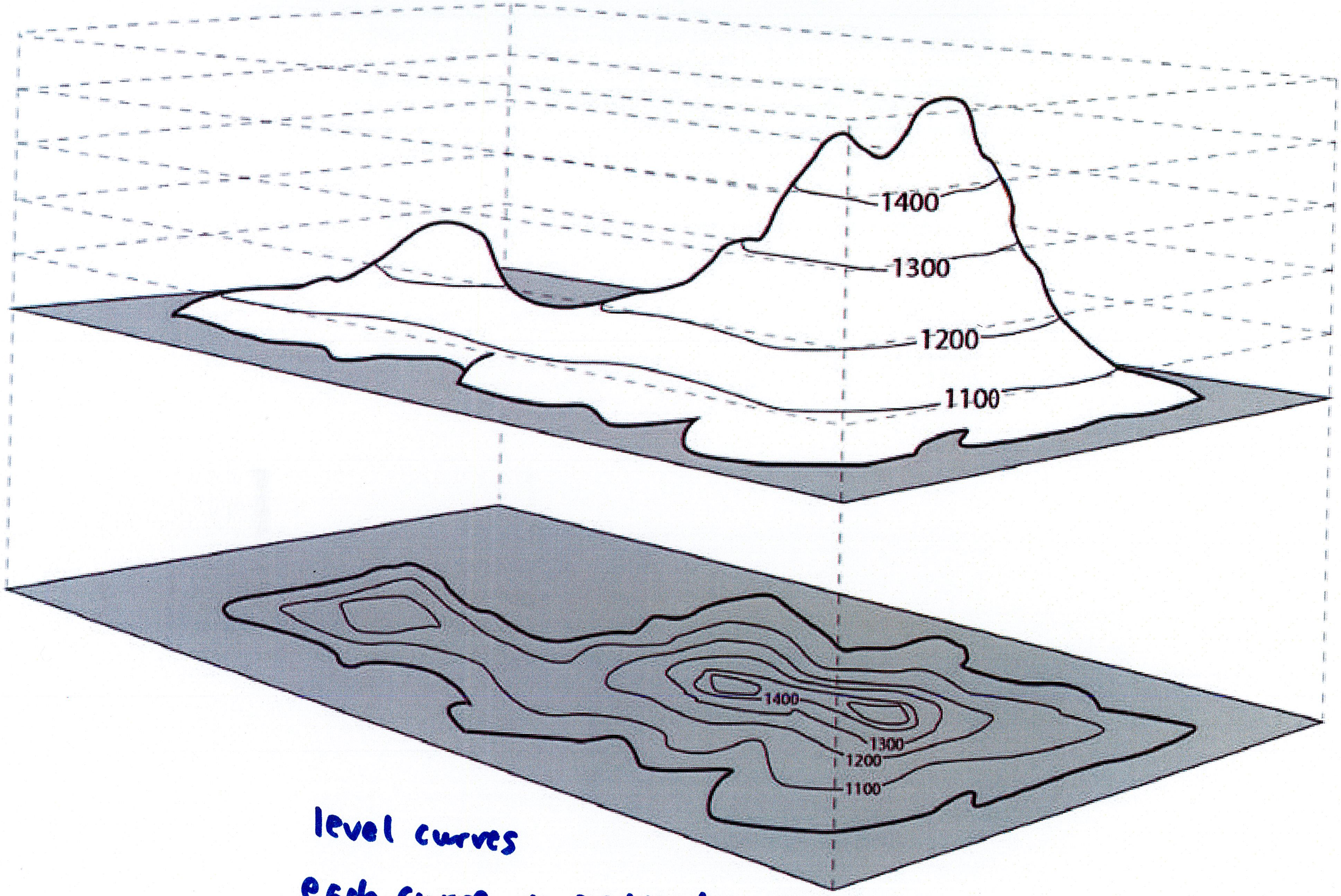


level curves of height of a hill

here, f_x tells us how fast the height changes if we walk in x direction (East)

f_y tells us how f changes if we go in y direction (North)

but what if we go in a direction such as North-East?



level curves

each curve → particular z

the directional derivative tells us how f changes given a direction

let $\vec{u} = \langle \underline{\underline{a}}, \underline{\underline{b}} \rangle$ be a unit vector

then the rate of change of $f(x,y)$ in the direction of $\vec{u}$ is

$$\boxed{D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+\underline{\underline{a}}h,\ y+\underline{\underline{b}}h) - f(x,y)}{h}}$$

note if $\vec{u} = \langle 1, 0 \rangle$ then $D_{\vec{u}} f(x,y) = f_x$

$\vec{u} = \langle 0, 1 \rangle$ " " $= f_y$

with some algebra, we can rewrite the formula as

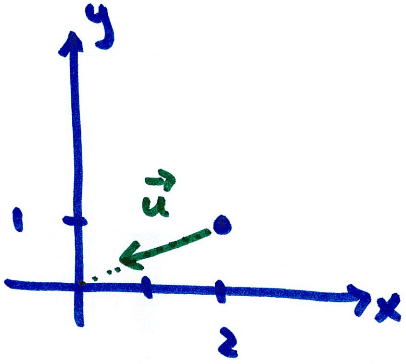
$$\boxed{\begin{aligned} D_{\vec{u}} f(x,y) &= \frac{\partial f}{\partial x}(x,y)\,a + \frac{\partial f}{\partial y}(x,y)\,b \\ &= f_x a + f_y b \end{aligned}}$$

where $\vec{u} = \langle a, b \rangle$ is the unit vector specifying direction

example $f(x,y) = \cos(2x+3y)$

find the rate of change of $f(x,y)$ at $(2,1)$ in the direction toward the origin.

$\vec{u}$ is very important



from $(2,1)$ to origin → $\langle -2, -1 \rangle$

$$\vec{u} = \frac{\langle -2, -1 \rangle}{|\langle -2, -1 \rangle|} = \frac{\langle -2, -1 \rangle}{\sqrt{5}}$$

$$= \left\langle \underset{a}{\frac{-2}{\sqrt{5}}}, \underset{b}{\frac{-1}{\sqrt{5}}} \right\rangle$$

$$D_{\vec{u}} f(x,y) = f_x(x,y)\, a + f_y(x,y)(b)$$

$f_x = -2\sin(2x+3y)$

$f_x(2,1) = -2\sin(7)$

$f_y = -3\sin(2x+3y)$

$f_y(2,1) = -3\sin(7)$

$$D_{\vec{u}} f(2,1) = -2\sin(7) \cdot \frac{-2}{\sqrt{5}} - 3\sin(7) \cdot \frac{-1}{\sqrt{5}} = \boxed{\frac{7}{\sqrt{5}}\sin(7)}$$

back to $D_{\vec{u}} f = f_x a + f_y b$

note this is also $$D_{\vec{u}} f = \underbrace{\langle f_x, f_y \rangle}_{\substack{\text{this vector}\\ \text{contains rates of change}}} \cdot \underbrace{\langle a, b \rangle}_{\vec{u}\text{ : unit vector giving direction}}$$

we call this vector the <u>Gradient</u>

$$\boxed{\begin{aligned} &\text{gradient of } f(x,y) : \quad \vec{\nabla} f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle \\ &\qquad\qquad \text{"del"} \qquad\qquad\quad = \frac{\partial f}{\partial x}(x,y)\,\vec{i} + \frac{\partial f}{\partial y}(x,y)\,\vec{j} \end{aligned}}$$

$\cancel{\vec{\nabla} f(x,y)} =$

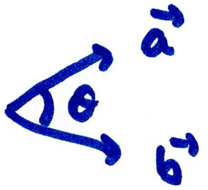
$$\boxed{D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}}$$

directional deriv : scalar projection onto $\vec{u}$

what does its magnitude and direction mean?

$$D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u}$$

recall: $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$



$$D_{\vec{u}} f = |\vec{\nabla} f| \overbrace{|\vec{u}|}^{1 \text{ because } \vec{u} \text{ is unit vector}} \cos\theta$$

$$= |\vec{\nabla} f| \underbrace{\cos\theta}_{-1 \leq \cos\theta \leq 1}$$

so, $-|\vec{\nabla} f| \leq D_{\vec{u}} f \leq |\vec{\nabla} f|$

$$\boxed{\text{this means the magnitude of the gradient is the maximum directional derivative}}$$

if $\vec{u}$ is same direction as $\vec{\nabla} f$

then $\theta = 0$ $\cos\theta = 1$

so $D_{\vec{u}} f = |\vec{\nabla} f|$ (max) → $\vec{u}$ is in direction of maximum or greatest ascent

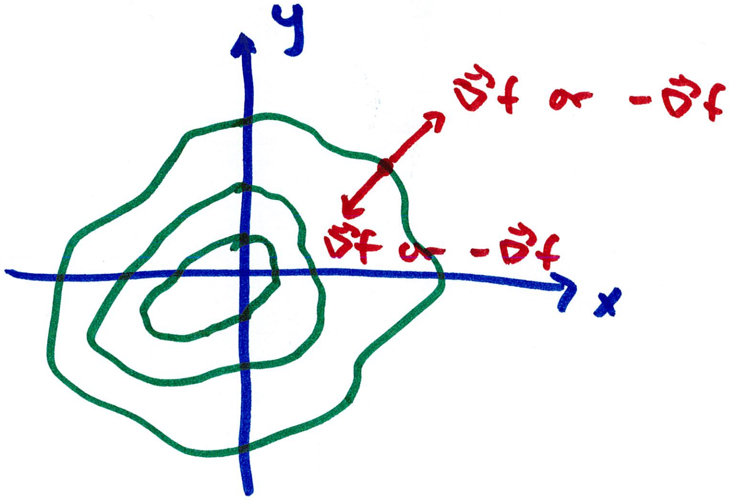
if $\vec{u}$ is opposite to $\vec{\nabla} f$, then $\vec{u}$ is direction of greatest descent note $D_{\vec{u}} f = \vec{\nabla} f \cdot \vec{u} = 0$ if $\vec{\nabla} f \perp \vec{u}$ if gradient is orthogonal to $\vec{u}$

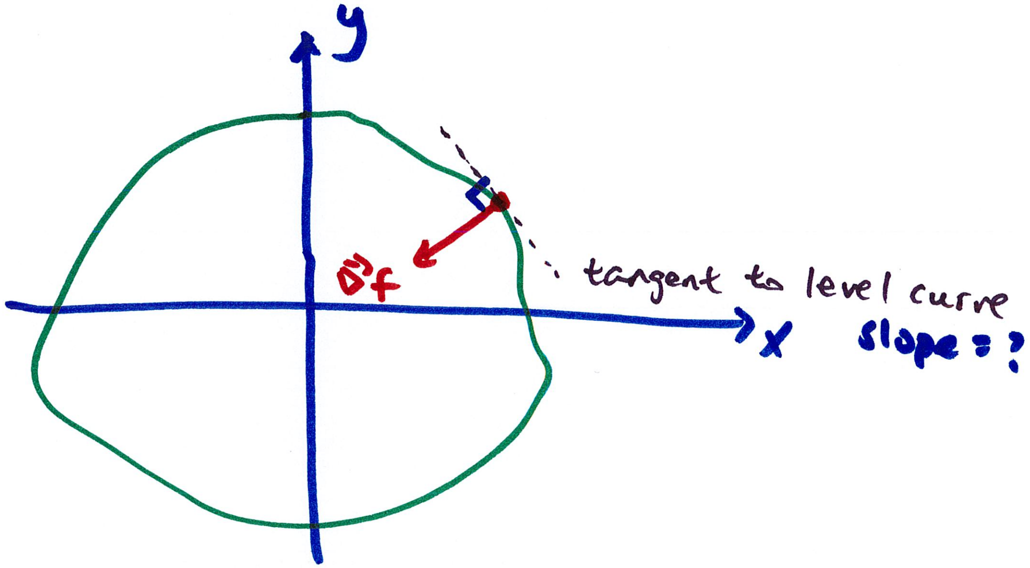
remember, a level curve is where $z = f(x,y) = \text{constant}$

so if $\vec{u}$ is along a level curve ($\vec{u}$ // tangent of level curve)

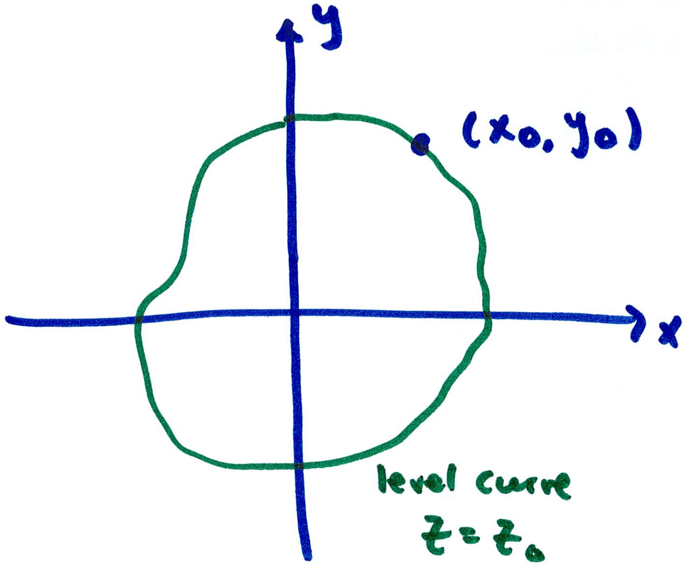
then $D_{\vec{u}} f = 0$

$$\boxed{\text{therefore, the gradient is orthogonal to a level curve}}$$





we can use the gradient and $D_{\vec{u}} f$ to find the slope of the level curve



we can "parametrize" the level curve as

$$\vec{r}(t) = \langle x(t), y(t) \rangle$$

at the point (x_0, y_0)

$$D_{\vec{u}} f(x_0, y_0) = \vec{\nabla} f(x_0, y_0) \cdot \vec{u}$$

If $\vec{u}$ is along a level curve then it's tangent to the level curve $\vec{r}(t)$

so, $\vec{u} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle}{|\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle|}$

along level curve, $D_{\vec{u}} f = 0$

so, $\langle f_x, f_y \rangle \cdot \dfrac{\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle}{|\langle \frac{dx}{dt}, \frac{dy}{dt} \rangle|} = 0 \rightarrow f_x \dfrac{dx}{dt} + f_y \dfrac{dy}{dt} = 0$

then, $\boxed{\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{dy}{dx} = -\dfrac{f_x}{f_y}}$ this is the slope of a level curve

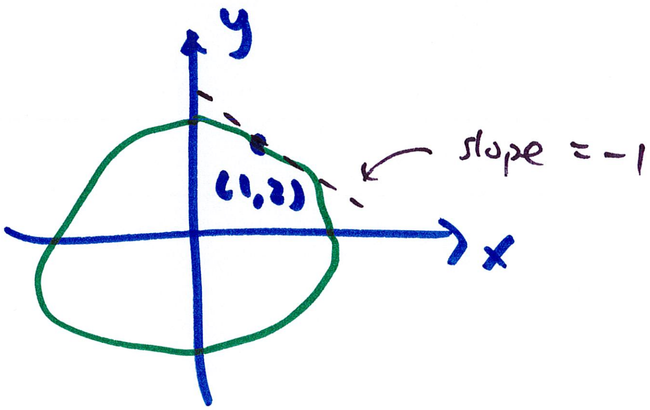
example $f(x,y) = xe^{y}$

find the slope of level curve at $x=1$, $y=2$

from last page, $\frac{dy}{dx} = -\frac{f_x}{f_y}$

$$= -\frac{e^{y}}{xe^{y}} = -\frac{1}{x}$$

at $(1,2)$, $\frac{dy}{dx} = -\frac{1}{1} = -1$



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