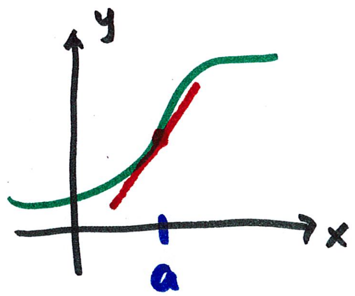
15.6 The Tangent Plane and Linear Approximation

NOT on exam 1. HW due Thu. 2/16

recall if $y = f(x)$, we can find the linear approx. of $f(x)$ near $x = a$

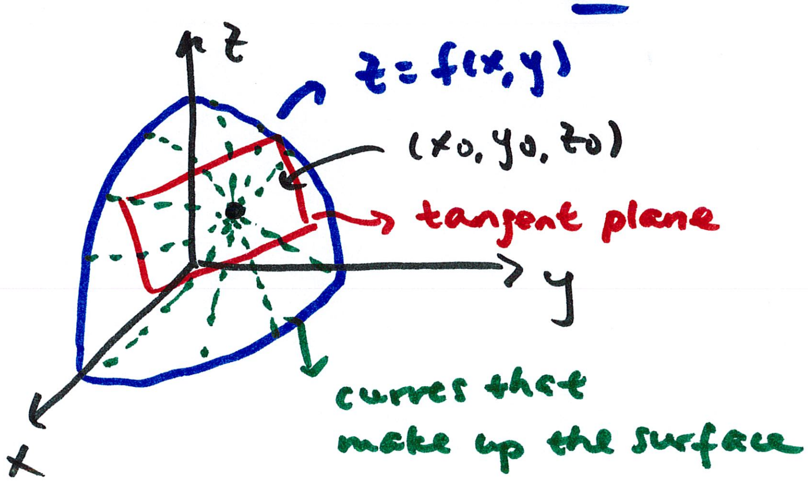


$$L \approx f(x) = f(a) + f'(a)(x-a)$$

as long as x is "near" a, $L \approx f(x)$

$z = f(x,y)$ is a surface and at (x_0, y_0, z_0), can we find the equivalent of the tangent line approx.?

→ tangent plane



the surface is made up of infinitely curves, each having a tangent vector at (x_0, y_0, z_0)

the tangent plane contains ALL of these tangent vectors

→ find vector $\perp$ to ALL tangent vectors the surface is made up of infinitely many curves $\rightarrow \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

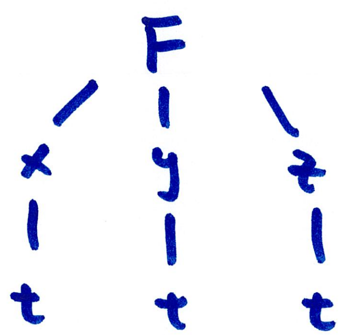
each having tangent vector $\underbrace{\vec{r}\,'(t) = \langle x'(t), y'(t), z'(t) \rangle}_{\text{find a vector } \perp \text{ to these}}$

Surface : $z = f(x, y)$

it contains many $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

so, $z = f(x, y) \rightarrow z(t) = f(x(t), y(t))$

defined a new function $\underbrace{F(x, y, z)}_{\text{really a function of } t} = f(x, y) - z = 0$



by the Chain Rule,

$$\frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

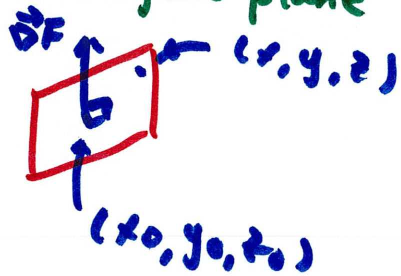
because $F = f - z = 0$

rewrite:

$$\underbrace{\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle}_{\substack{\text{gradient of } F \\ \vec{\nabla} F}} \cdot \underbrace{\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dz} \right\rangle}_{\text{tangent vectors on surface}} = 0$$

So, $\vec{\nabla} F = \langle F_x, F_y, F_z \rangle$ is $\perp$ to all tangent vectors (which form the tangent plane)

the gradient of $\vec{F} = f - z$ is the <u>normal vector</u> of the tangent plane



$$\boxed{\begin{array}{l} \langle F_x, F_y, F_z \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \\ \text{or} \\ F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0 \\ \qquad \text{where } F = f - z \end{array}}$$

if we prefer to use $z = f(x,y)$ explicitly, then make a minor tweak

$$F = f - z \qquad \text{so } F_x = f_x \,,\ F_y = f_y \,,\ F_z = -1$$

$$\boxed{\begin{gathered} f_x(x-x_0) + f_y(y-y_0) - (z-z_0) = 0 \\ \text{or} \quad z - z_0 = f_x(x-x_0) + f_y(y-y_0) \end{gathered}}$$

<u>example</u> $z = f(x,y) = \sqrt{x^2+y^2}$ cone (upper half)

find the tangent plane at $(3,4,5)$

define $F = f(x,y) - z = \sqrt{x^2+y^2} - z$

$$\vec{\nabla} F = \langle F_x, F_y, F_z \rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1 \right\rangle$$

$$\vec{\nabla} F(3,4,5) = \left\langle \frac{3}{5}, \frac{4}{5}, -1 \right\rangle$$

$$\boxed{\begin{gathered} \text{tangent plane: } \left\langle \frac{3}{5}, \frac{4}{5}, -1 \right\rangle \cdot \langle x - \overset{x_0}{3}, y - \overset{y_0}{4}, z - \overset{z_0}{5} \rangle = 0 \\ \frac{3}{5}(x-3) + \frac{4}{5}(y-4) - (z-5) = 0 \end{gathered}}$$

near $(3,4,5)$ we expect the tangent plane $\approx$ true surface function

$$\text{Tp}: \ z = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$$

which should be close to true $z = f(x,y) = \sqrt{x^2+y^2}$

let's try $x = 3.01$, $y = 3.99$

TP approx: $z = 5 + \frac{3}{5}(0.01) + \frac{4}{5}(-0.01) = 4.998$

true value of z: $f(3.01, 3.99) = \sqrt{(3.01)^2 + (3.99)^2} = 4.99802$

they are close because Δx, Δy are small tangent plane : $z - z_0 = f_x \underbrace{(x - x_0)}_{\Delta x} + f_y \underbrace{(y - y_0)}_{\Delta y}$

let $x - x_0 = \Delta x = dx$ (very small change)

$y - y_0 = dy$

$z - z_0 = dz$

then : $dz = f_x \, dx + f_y \, dy$ this tells us the change in z given changes in x and y

<u>example</u> $z = f(x,y) = x^2 y$

x starts at 1 and ends at $1.01 \rightarrow dx = 0.01$

y " " 3 " " $2.91 \rightarrow dy = -0.09$

find approx change in $z \rightarrow dz = ?$

$dz = f_x \, dx + f_y \, dy$

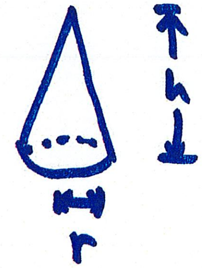
$$dz = (2xy)\underbrace{dx}_{0.01} + (x^2)\underbrace{dy}_{-0.09}$$

use the starting x, y because that's where the tangent plane is formed

$$= (2\cdot 1\cdot 3)(0.01) + (1)^2(-0.09) = -0.03$$

this is the estimate of change in $z = x^2y$ given $dx = 0.01$ and $dy = -0.09$

example The volume of a cone is $V = \frac{1}{3}\pi r^2 h$



if r is increased by 1%

and h is decreased by 3%

what is the approx. % change in volume V?

$$V = \frac{1}{3}\pi r^2 h$$

$$dV = \frac{\partial V}{\partial r}\,dr + \frac{\partial V}{\partial h}\,dh$$

$(dz = f_x\,dx + f_y\,dy)$

$$dV = \left(\frac{2}{3}\pi r h\right)dr + \left(\frac{1}{3}\pi r^2\right)dh$$

refer to absolute change, NOT relative change

DO NOT put in 1% for dr and -3% for dh

divide the eq. by $V = \frac{1}{3}\pi r^2 h$

relative (%) changes

relative change in V

$$\frac{dV}{V} = \frac{\frac{2}{3}\pi r h}{\frac{1}{3}\pi r^2 h}\,dr + \frac{\frac{1}{3}\pi r^2}{\frac{1}{3}\pi r^2 h}\,dh = 2\frac{dr}{r} + \frac{dh}{h}$$

now we get

$$\frac{dV}{V} = 2(0.01) + (-0.03) = -0.01 = 1\% \text{ decrease}$$

$0.01 \downarrow$ 1% increase

$-0.03 \downarrow$ 3% decrease