15.7 Max and Min Problems (part 1)

recall if $y = f(x)$, if $f'(c) = 0 \rightarrow$ c is a critical number

$(c, f(c))$ is a critical point

$x = c$ is a possible location of relative max/min of $f(x)$

then the Second Derivative Test says if $\underbrace{f''(c) > 0}_{\text{concave up, so like } \cup} \rightarrow$ rel. min at $x = c$

if $\underbrace{f''(c) < 0}_{\text{concave down, so like } \cap} \rightarrow$ rel. max at $x = c$

if $f''(c) = 0 \rightarrow$ inconclusive

for $z = f(x, y)$, a lot of this carry over, but some things change the two-variable version of <u>Second Derivative Test</u> :

find critical pts : $\underbrace{f_x = 0 \ \underline{\text{AND}} \ f_y = 0}_{\text{collect critical pts } (a,b)}$

then we evaluate the <u>discriminant</u>

$$D = f_{xx} f_{yy} - (f_{xy})^2$$

if at (a,b) $\underline{f_{xx} > 0 \text{ and } D > 0}$ → rel <u>min</u> at (a,b)

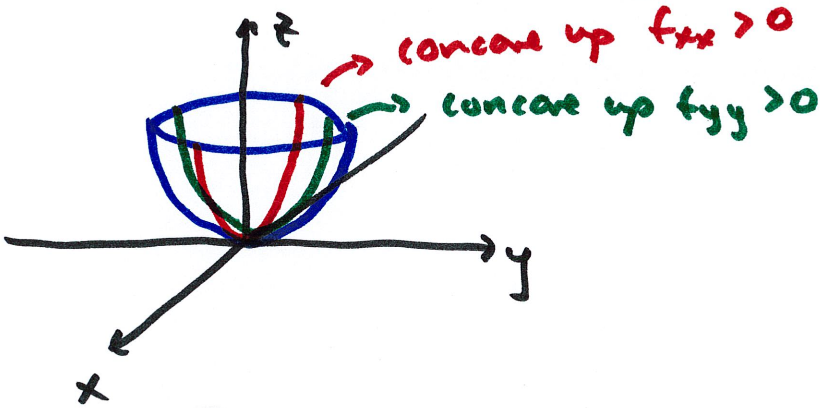
$\underline{f_{xx} < 0 \text{ and } D > 0}$ → rel. <u>max</u> at (a,b)

$\underline{D < 0}$ → <u>saddle point</u> at (a,b)
(neither max nor min)

$\underline{D = 0}$ → <u>inconclusive</u>

D contains shape information near a critical point

for example, $f(x,y) = x^2 + y^2 \rightarrow \underline{f_{xx} = 2}, \ \underline{f_{yy} = 2}, \ f_{xy} = 0$

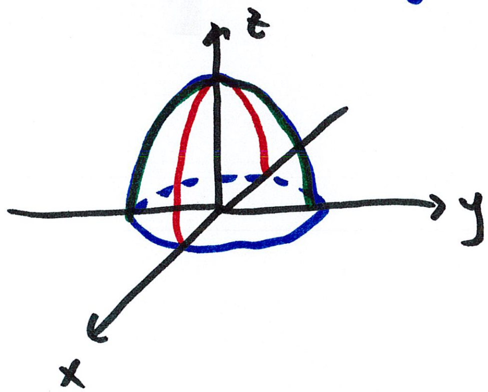


at $(0,0)$ $D = f_{xx} f_{yy} - (f_{xy})^2 > 0$

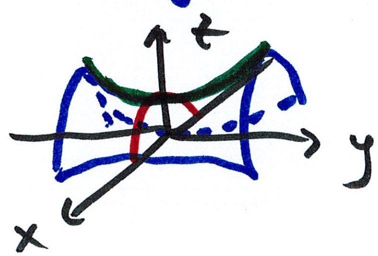
(rel. min)

another example, $f(x,y) = 4 - x^2 - y^2 \rightarrow \underline{f_{xx} = -2}, \ \underline{f_{yy} = -2} \ \ f_{xy} = 0$

at $(0,0)$ $D > 0$



last example $f(x,y) = -x^2 + y^2 \rightarrow \underbrace{f_{xx} = -2}_{\text{at } (0,0)}$ (concave down) $\underbrace{f_{yy} = 2}_{D < 0}$ (concave up) $f_{xy} = 0$



$D > 0 \rightarrow$ near crit. pt parabolic approx

$D < 0 \rightarrow$ can't do that

example $f(x,y) = x^3 - 48xy + 64y^3$

find critical pts : $f_x = 0$ AND $f_y = 0$

$f_x = 3x^2 - 48y + 0 = 0$ — ①

$f_y = -48x + 192y^2 = 0$ — ②

from ① $x^2 = 16y$

from ② $x = 4y^2$

$(4y^2)^2 = 16y$

$16y^4 - 16y = 0$

$16y(y^3 - 1) = 0$

$\underbrace{y = 0 \text{ or } y = 1}_{\text{these are the y-coord of critical pts}}$

then from $x = 4y^2$, we get the following crit. pts

$$\boxed{(0,0)\ ,\ (4,1)}$$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$f_{xx} = 6x \quad f_{yy} = 384y \quad f_{xy} = f_{yx} = -48$

$D = 2304xy - 2304$

evaluate at critical pts

$D(0,0) < 0 \longrightarrow$ saddle pt at $(0,0)$

$D(4,1) > 0 \quad f_{xx}(4,1) > 0$ rel. min at $(4,1)$

example $f(x,y) = xye^{-x^2-y^2}$

find crit. pts : $f_x = 0$ AND $f_y = 0$

$f_x = \cdots = ye^{-x^2-y^2}(-2x^2+1) = 0$ ①

$f_y = \cdots = xe^{-x^2-y^2}(-2y^2+1) = 0$ ②

from ① $\underbrace{(e^{-x^2-y^2})}_{\text{exponential is never } 0}\underbrace{(y)(-2x^2+1)}_{\text{these are where crit. pts come from}} = 0$

so, $\boxed{y = 0}$ or $\boxed{x = \pm\frac{1}{\sqrt{2}}}$

do NOT pair these to form points because these came out of ① $= 0$ so forming points with these will NOT guarantee ② $= 0$

from ② $(e^{-x^2-y^2})(x)(2y^2-1)=0$

so, $\boxed{x=0}$ or $\boxed{y=\pm\frac{1}{\sqrt{2}}}$

pick one black boxed value from each to form crit. pts :

$(0,0)$, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$f_{xx} = 2xy(2x^2-3)e^{-x^2-y^2}$

$f_{yy} = 2xy(2y^2-3)e^{-x^2-y^2}$

$f_{xy} = (2x^2-1)(2y^2-1)e^{-x^2-y^2}$

at $(0,0)$ $D<0$

at $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ $D>0$, $f_{xx} \cancel{>} 0$ rel. ~~min~~ max ~~rel. min~~ ~~rel. max~~

at $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$ $D>0$, $f_{xx} \cancel{<} 0$ rel ~~max~~ min what if $D = 0$?

<u>example</u> $f(x,y) = xy^2$

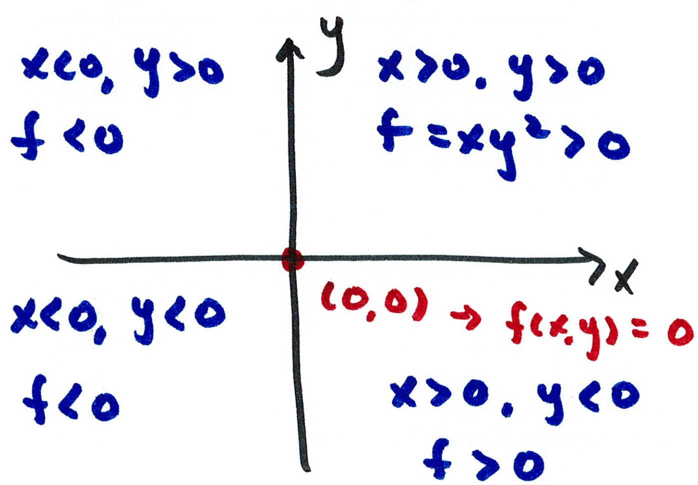
$(0,0)$ is a crit. pt

$f_x = y^2$, $f_y = 2xy$, $f_{xy} = 2y$, $f_{yy} = 2x$ $f_{xx} = 0$

$D = f_{xx} f_{yy} - (f_{xy})^2 = -4y^2$

at $(0,0)$ $D = 0$ inconclusive : crit. pt can still be max/min/saddle pt

one way to tell is to see the signs of $f(x,y) = xy^2$ near $(0,0)$



as we move right, f increases

as we move left f decreases

so $(0,0)$ is NOT a max or min