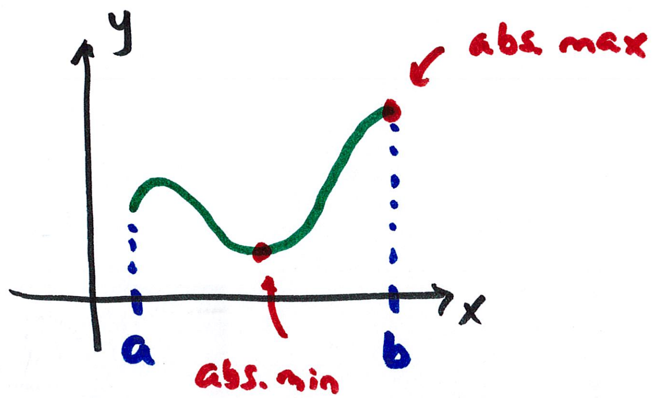
15.7 Max/min problems (part 2)

if $y = f(x)$, $a \le x \le b$ then the absolute max/min of $f(x)$ can be at the ends of the interval or at critical pts inside the interval



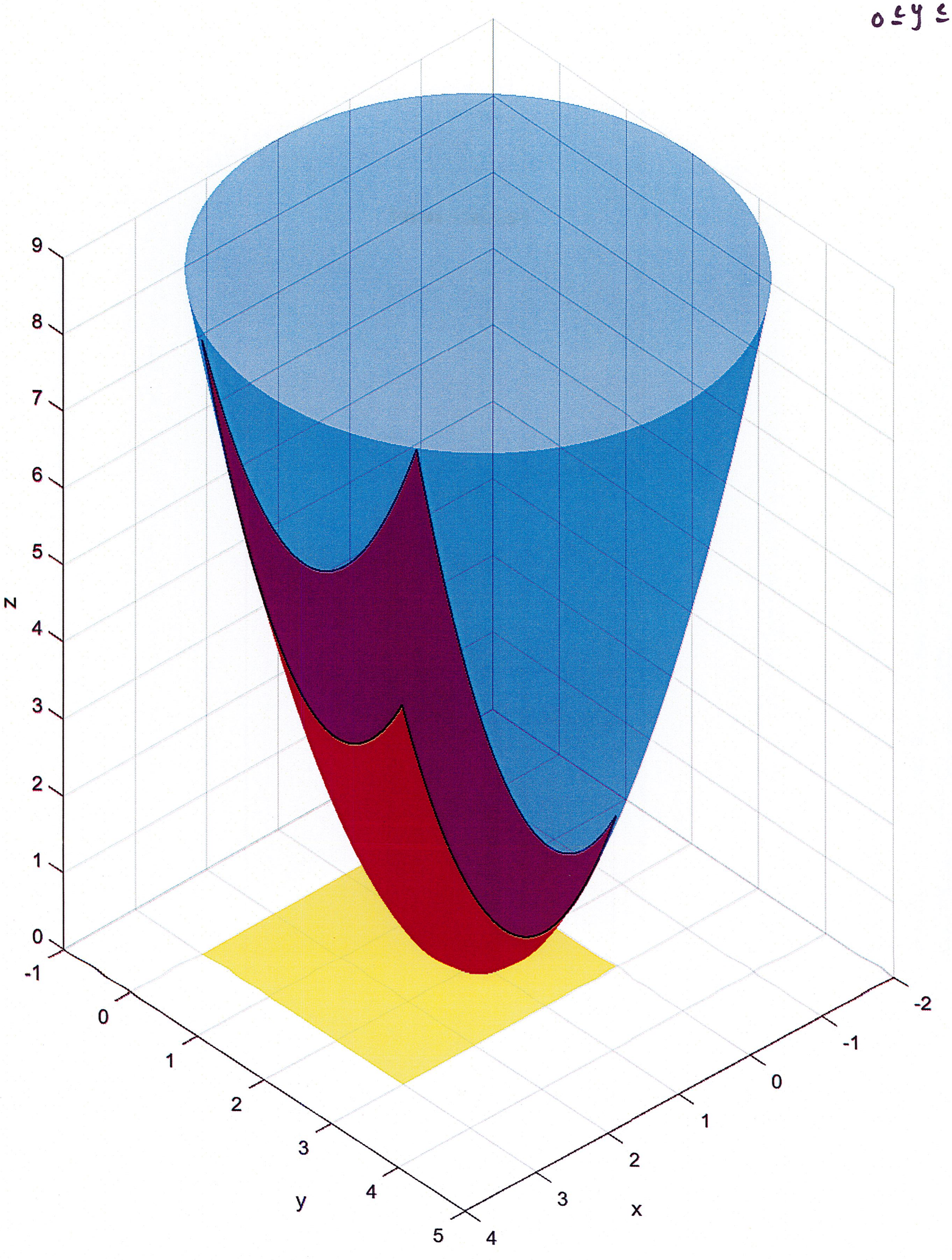
procedure: find all interior critical pts
compare $f(x)$ at the critical pts and at ends to find max/min

$z = f(x,y)$ is a surface and domain is a region in xy-plane and the boundaries are curves

the basic idea is the same : find all interior critical pts
then find all possible locations of max/min on the boundaries
then compare $f(x,y)$

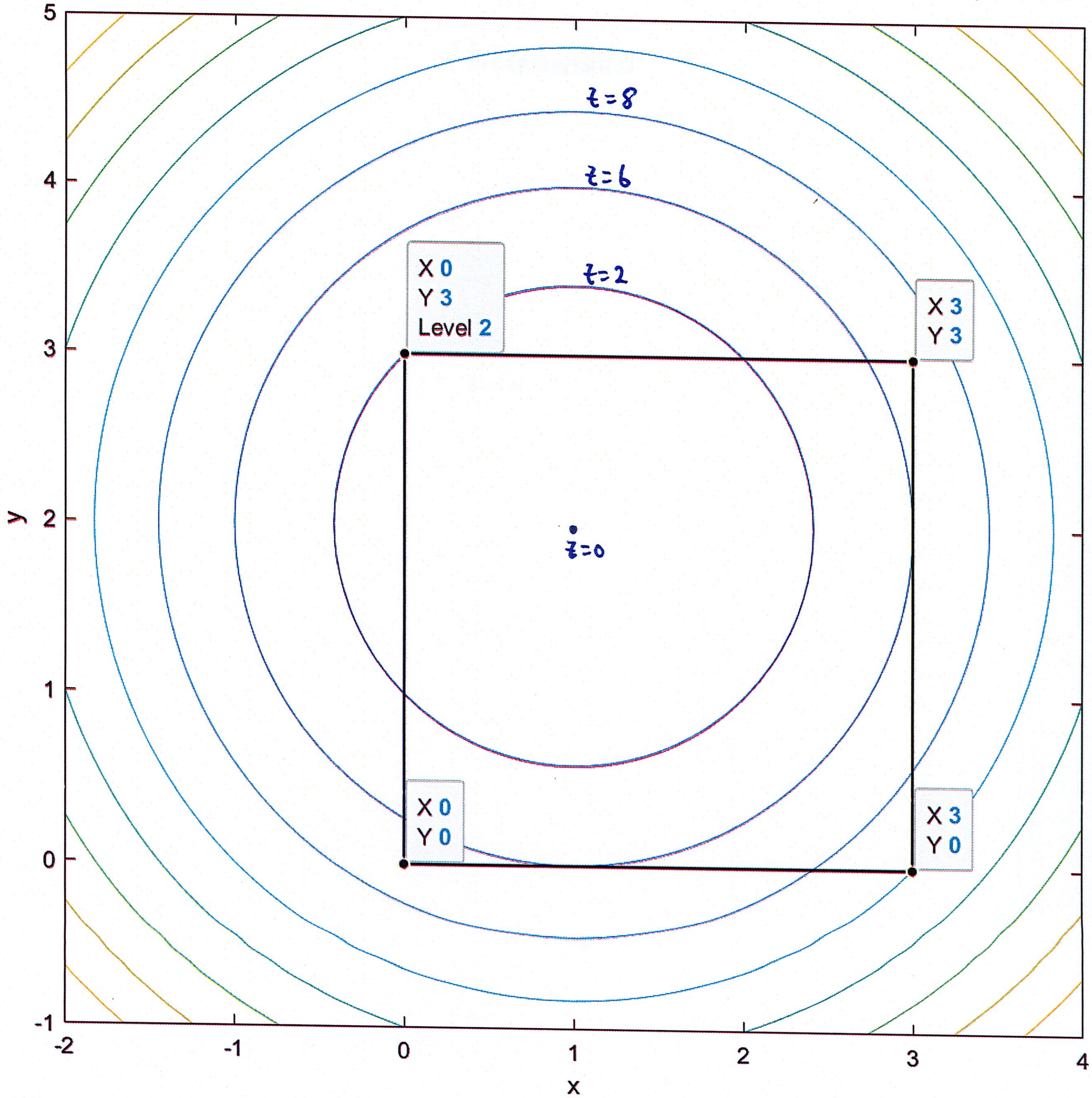
find max/min of

$z = f(x,y) = (x-1)^2 + (y-2)^2$ on $0 \le x \le 3$, $0 \le y \le 3$



find max/min of

$z = f(x,y) = (x-1)^2 + (y-2)^2$ on $0 \le x \le 3$, $0 \le y \le 3$



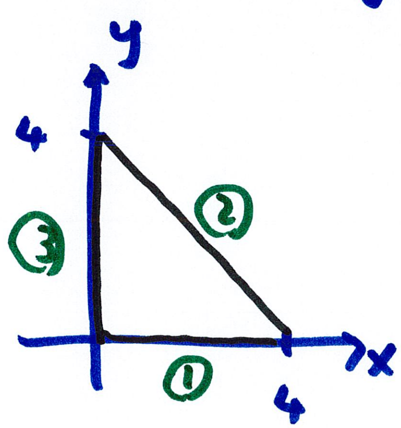
example $z = f(x,y) = x^2 + y^2 - 2x - 4y + 10$

find max/min above the triangle with vertices $(0,0)$, $(0,4)$, $(4,0)$

paraboloid $z = f(x,y) = (x-1)^2 + (y-2)^2 + 5$

find the max/min of $f(x,y)$ inside the triangle and on the boundary of the triangle

notice the triangle can be broken down into 3 parts



on ① $0 \le x \le 4$, $y = 0$

② $y = 4 - x$, $0 \le x \le 4$

③ $0 \le y \le 4$, $x = 0$

find critical pts inside triangle

then locations of max/min on each edge

then compare

find interior critical pts : $f_x = 0$, $f_y = 0$

$$\left.\begin{aligned} f_x &= 2x-2 = 0 \\ f_y &= 2y-4 = 0 \end{aligned}\right\} \quad \boxed{cp : (1,2)}$$

is this inside the triangle? yes
so we keep it

now we look at 1

$y=0, \quad 0 \le x \le 4$

$f(x,y) = x^2+y^2-2x-4y+10$ becomes $f(x) = x^2-2x+10, \quad 0 \le x \le 4$

notice this is reduced to a one-variable abs max/min prob.

critical pts : $f'(x) = 2x-2 = 0 \rightarrow x=1, \ y=0 \rightarrow \boxed{(1,0)}$

ends : $x=0, \ y=0 \rightarrow \boxed{(0,0)}$

$x=4, \ y=0 \rightarrow \boxed{(4,0)}$

on ② $y = 4 - x$, $0 \le x \le 4$

$f(x,y) = x^2 + y^2 - 2x - 4y + 10$ becomes

$f(x) = 2x^2 - 6x + 10$, $0 \le x \le 4$ again, a one-variable abs. max/min

$f'(x) = 4x - 6 = 0 \rightarrow x = 3/2$, $y = 4 - x = 4 - 3/2 = 5/2$

$\boxed{(3/2, 5/2)}$

ends: $x = 0$, $y = 4 - x \rightarrow \boxed{(0, 4)}$

$x = 4$, $y = 4 - x \rightarrow \boxed{(4, 0)}$

on ③ $x = 0$, $0 \le y \le 4$

$f(x,y) = x^2 + y^2 - 2x - 4y + 10$ becomes

$f(y) = y^2 - 4y + 10$ $0 \le y \le 4$

following the same procedure as on ① and ② we get

the following points of interest : $\boxed{(0,2),\ (0,0),\ (0,4)}$

now we compare $f(x,y) = x^2+y^2-2x-4y+10$ at each of those points of interest

$f(1,2) = 5 \rightarrow$ abs. min at $x=1$, $y=2$, $z=5$ (vertex)

$f(0,0) = 10$

$f(4,0) = 18 \rightarrow$ abs. max at $x=4$, $y=0$, $f(x,y) = 18$

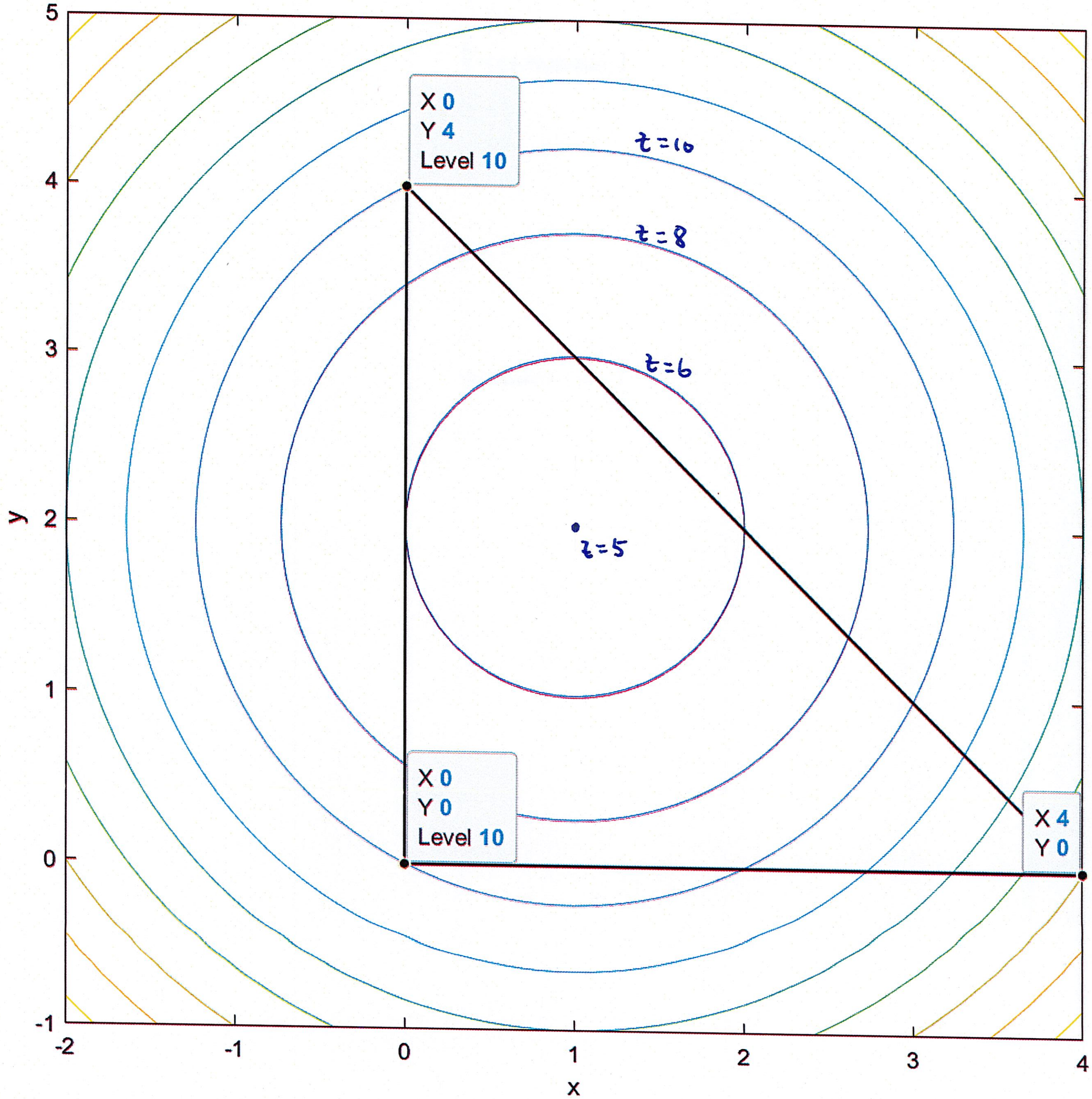
$f(0,4) = 10$

$f(3/2, 5/2) = 11/2$

$f(0,2) = 6$

$f(1,0) = 9$

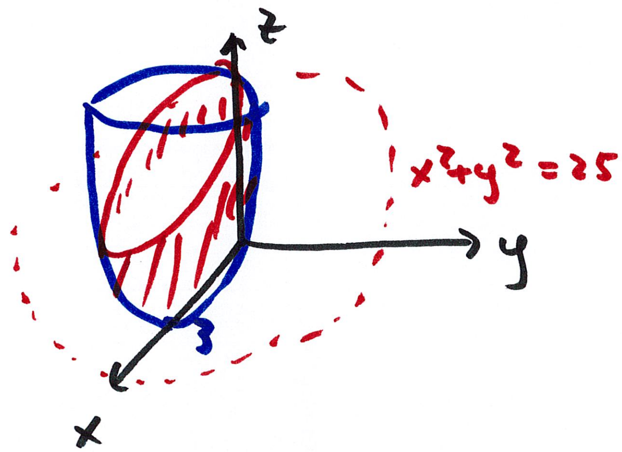
$z = f(x,y) = x^2 + y^2 - 2x - 4y + 10$

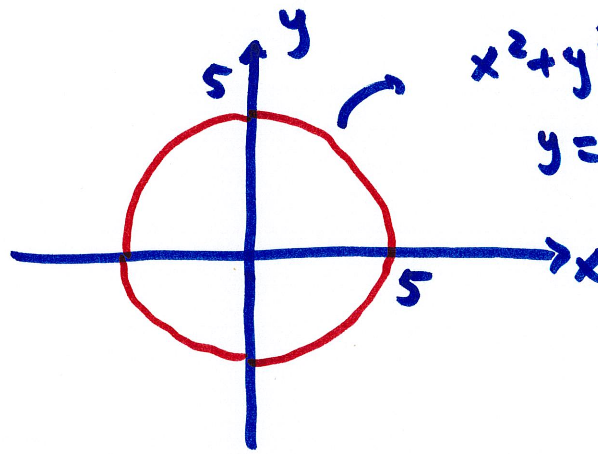


example Find abs max/min of $f(x,y) = x^2+y^2-6x+9$ → paraboloid

above the region $x^2+y^2 \le 25$ → circle

find highest/lowest on the paraboloid above the circle





$x^2+y^2=25$

$y = \sqrt{25-x^2}$ (upper)

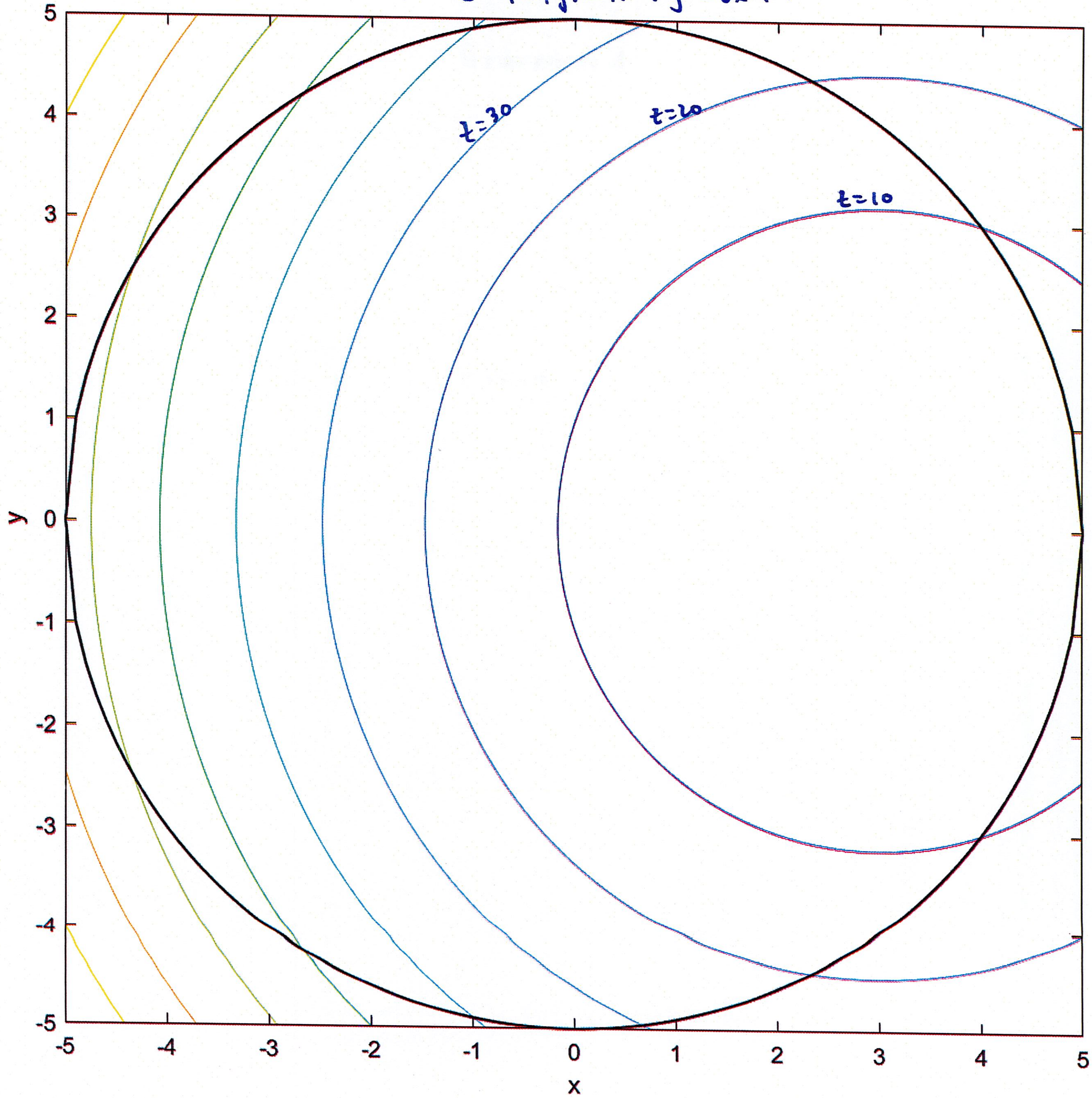
$y = -\sqrt{25-x^2}$ (lower)

$f(x,y) = x^2+y^2-6x+9$

$$\left.\begin{aligned} f_x &= 2x-6 = 0 \\ f_y &= 2y = 0 \end{aligned}\right\} \quad \boxed{cp: (3,0)}$$

is this inside the restricted area?
yes, so we keep it

$z = f(x,y) = x^2 + y^2 - 6x + 9$



now let's travel along $\underbrace{y = \sqrt{25-x^2} \text{ and } y = -\sqrt{25-x^2}, \ -5 \le x \le 5}_{x^2+y^2=25}$

$$f(x,y) = \underbrace{x^2+y^2}_{25} - 6x + 9$$

$f(x) = 34 - 6x \qquad -5 \le x \le 5$

$f' = -6$ no critical pt

ends: $x = -5, \ y = 0 \rightarrow \boxed{(-5, 0)}$

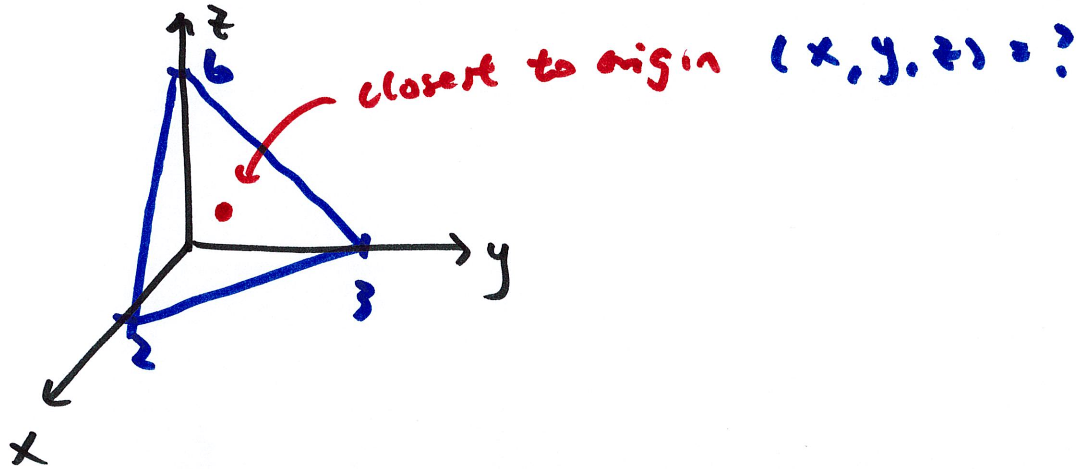
$x = 5, \ y = 0 \rightarrow \boxed{(5, 0)}$

compare: $f(3, 0) = 0$ abs. min

$f(-5, 0) = 64$ abs. max

$f(5, 0) = 4$

example what point on the plane $3x+2y+z=6$ is closest to the origin?



closest to origin: minimize distance

$$d=\sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$$

minimize this

$$=\sqrt{x^2+y^2+z^2}$$

while $3x+2y+z=6$

or $z=6-3x-2y$

$$=\sqrt{x^2+y^2+(6-3x-2y)^2}$$

since distance ≥ 0 $\rightarrow$ minimize d^2 is ~~de~~ equivalent

so minimize $f(x,y)=d^2=x^2+y^2+(6-3x-2y)^2$

find cp: $f_x=0$, $f_y=0$ then use 2nd Deriv. Test