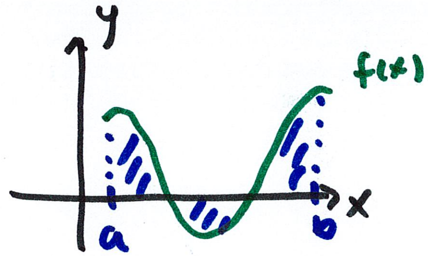
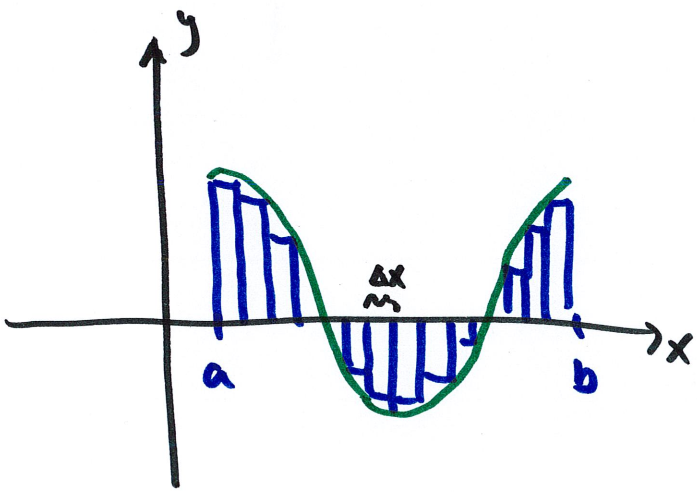
# 16.1 Double Integrals over Rectangular Regions

if $y = f(x)$, $a \le x \le b$




then $\int_a^b f(x)\,dx$ gives us the net area between $f(x)$ and the x-axis

but remember $\int_a^b f(x)\,dx$ really sums up infinitely-many rectangle areas




total area = sum of rectangles

each rectangle has height $f(x_i)$

sample point (left/right/mid point)

then $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i)\Delta x$

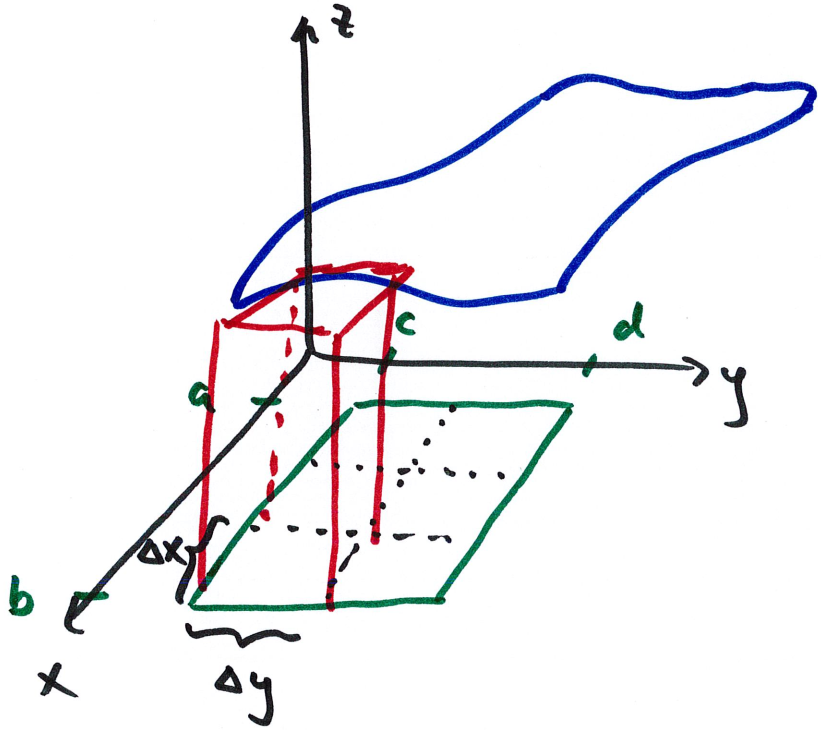
\# of rectangles

$= \int_a^b f(x)\,dx$

$z = f(x,y)$ is a surface

if we want the volume under the surface above $\overbrace{a \le x \le b,\quad c \le y \le d}^{[a,b]\times[c,d]}$

we can use the same idea as in one-variable case




we want volume under $f(x,y)$
above the rectangle

subdivide $x$ into $n$ parts
$y$ " $m$ parts

use a particular choice of
sample points to find height
of a rectangular box

the rectangle box has volume $f(x_i, y_j)\overbrace{\Delta x \Delta y}^{\Delta A}$

then sum up all $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i, y_j)\,\Delta A$

now let $n \to \infty$, $m \to \infty$, $\Delta x \to dx$ $\Delta y \to dy$

$$\lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\,\Delta x \Delta y \qquad a \le x \le b \qquad c \le y \le d$$

$$= \underbrace{\int_a^b \underbrace{\int_c^d f(x,y)\,dy}_{y}\,dx}_{x} = \int_a^b \int_c^d f(x,y)\,dA$$

volume of box: $f(x,y)\,dy\,dx = f(x,y)\,dx\,dy$

$$\underbrace{\int_c^d \underbrace{\int_a^b f(x,y)\,dx}_{x}\,dy}_{y}$$

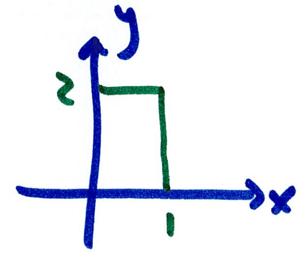
if the region is rectangular ( $a \le x \le b$, $c \le y \le d$ ) then the order does NOT matter

## example

$$\int_0^1 \int_0^2 (3-x-y)\,dy\,dx$$

$0 \le y \le 2$

$0 \le x \le 1$




basic idea: work inside-out

inner integral:

$$\int_0^2 (3-x-y)\,dy$$

"constant"

"live" variable

others are constants

$$= 3y - xy - \frac{1}{2}y^2 \Big|_{y=0}^{y=2} = 6 - 2x - 2 = 4 - 2x$$

outside integral:

$$\int_0^1 (4-2x)\,dx = 4x - x^2 \Big|_{x=0}^{x=1} = \boxed{3}$$

this is rectangular region, order doesn't matter

$$\int_0^2 \int_0^1 (3-x-y)\,dx\,dy$$

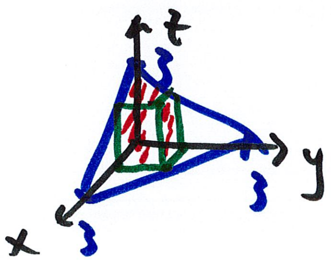
inside: $\displaystyle\int_0^1 (3-x-y)\,d\underline{x}$

↳ others are constants

$$= 3x - \tfrac{1}{2}x^2 - yx \Big|_{x=0}^{x=1} = 3 - \tfrac{1}{2} - y = \tfrac{5}{2} - y$$

outside: $\displaystyle\int_0^2 \left(\tfrac{5}{2} - y\right)dy = \tfrac{5}{2}y - \tfrac{1}{2}y^2 \Big|_{y=0}^{y=2} = 5 - 2 = \boxed{3}$

interpretation: volume under $z = 3-x-y$ above $0 \le x \le 1$, $0 \le y \le 2$



example

$$\int_0^1 \int_0^2 y^5 x^2 e^{x^3 y^3}\, dx\, dy$$

inside : $\displaystyle\int_0^2 y^5 x^2 e^{x^3 y^3}\, d\underline{x}$

↳ y is constant

$$= y^5 \int_0^2 x^2 e^{y^3 x^3}\, dx$$

sub: $u = y^3 x^3$

$du = 3y^3 x^2\, dx$

$x^2\, dx = \frac{1}{3y^3}\, du$

$$= y^5 \int_{x=0}^{x=2} e^u \cdot \frac{1}{3y^3}\, du = \frac{y^5}{3y^3} \int_{x=0}^{x=2} e^u\, du$$

$$= \frac{1}{3} y^2 \left( e^u \Big|_{x=0}^{x=2} \right. = \frac{1}{3} y^2 \left( e^{y^3 x^3} \Big|_{x=0}^{x=2} \right)$$

$$= \frac{1}{3} y^2 e^{8y^3} - \frac{1}{3} y^2$$

outside: $$\int_0^1 \left(\frac{1}{3}y^2e^{8y^3} - \frac{1}{3}y^2\right)dy$$

$$= \underbrace{\int_0^1 \frac{1}{3}y^2e^{8y^3}\,dy}_{\substack{u=8y^3 \\ du=24y^2dy \\ \vdots}} - \underbrace{\int_0^1 \frac{1}{3}y^2\,dy}_{\text{easy}} = \cdots = \boxed{\frac{1}{72}e^8 - \frac{1}{8}}$$

what if we had reversed the order?

$$\int_0^2 \int_0^1 y^5x^2e^{x^3y^3}\,dy\,dx$$

inside: $$\int_0^1 y^5x^2e^{x^3y^3}\,dy$$

x is constant

$$= x^2\int_0^1 y^5e^{x^3y^3}\,dy$$

by parts $uv - \int v\,du$

$u = y^5 \quad dv = e^{x^3y^3}dy$

$\vdots$

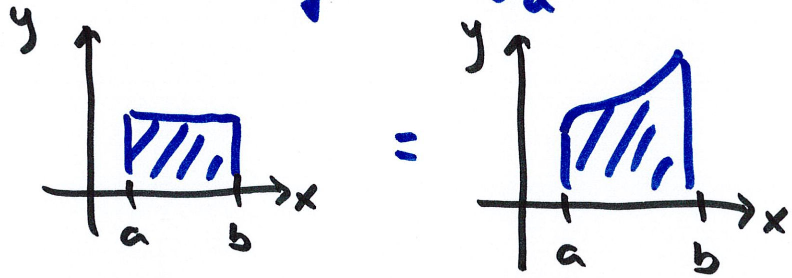
etc

recall if $y = f(x)$, $a \le x \le b$

then the average value of $f(x)$ over $a \le x \le b$ is

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

because $(b-a) f_{avg} = \int_a^b f(x)\,dx$



same idea for $z = f(x,y)$ $a \le x \le b$ $c \le y \le d$

$$f_{avg} = \frac{1}{A} \int_a^b \int_c^d f(x,y)\,dy\,dx = \frac{1}{A} \int_c^d \int_a^b f(x,y)\,dx\,dy$$

area of $[a,b] \times [c,d]$

$$= \frac{1}{A} \iint_R f(x,y)\,dA$$

region we integrate over

example

$$\int_0^1 \int_0^{\pi/3} x^2 \cos(xy)\,dx\,dy$$

order as given, inside : $\displaystyle\int_0^{\pi/3} x^2 \cos(xy)\,d\underline{\underline{x}}$ → $y$ is constant

two rounds of integration by parts

not "easy"

try switching : $\displaystyle\int_0^{\pi/3} \int_0^1 x^2 \cos(xy)\,dy\,dx$

inside : $\displaystyle\int_0^1 x^2 \cos(xy)\,d\underline{\underline{y}}$ $x$ const

$$= x^2 \sin(xy) \cdot \frac{1}{x} \Big|_{y=0}^{y=1} = x \sin x$$

outside : $\displaystyle\int_0^{\pi/3} x \sin x \, dx$ one round of by parts