16.2 Double Integrals over General Regions

last time : over rectangular regions $R = \{(x,y) : a \le x \le b, \quad c \le y \le d\}$

the order of integration doesn't matter

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

but switching order arbitrarily is only ok for rectangular regions

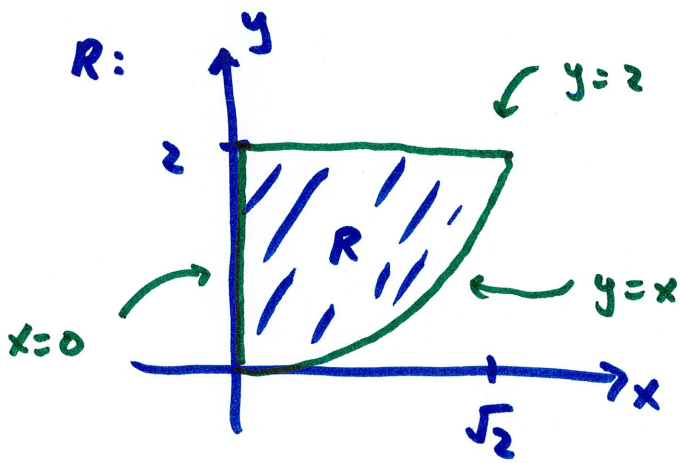
for general regions, order is very crucial

<u>example</u> $f(x,y) = xy^2$ $\quad R = \{(x,y) : \quad 0 \le x \le \sqrt{2}, \quad x^2 \le y \le 2\}$

top curve $y=2$

bottom curve $y = x^2$

R:



$$\boxed{\text{Basic rule : integrate the constant-bounded variable LAST (outside)}}$$

So, here we integrate x last, y first

$$\int_0^{\sqrt{2}} \int_{x^2}^{2} xy^2 \, dy \, dx$$

then proceed as before: inside-out

$x \qquad y$

$$= \int_0^{\sqrt{2}} \left(\frac{1}{3} x y^3 \Big|_{y=x^2}^{y=2} \right) dx = \int_0^{\sqrt{2}} \left(\frac{8}{3} x - \frac{1}{3} x^7 \right) dx$$

$$= \frac{4}{3} x^2 - \frac{1}{24} x^8 \Big|_0^{\sqrt{2}} = \boxed{2}$$

what if we integrated using the wrong order?

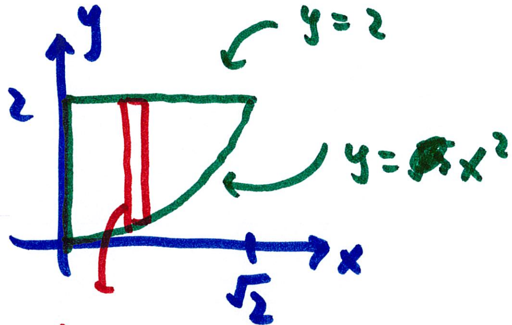
$$\int_{x^2}^{2} \int_0^{\sqrt{2}} xy^2 \, dx \, dy = \int_{x^2}^{2} \frac{1}{2} x^2 y^2 \Big|_{x=0}^{x=\sqrt{2}} dy$$

what do we do with this?

$$= \int_{x^2}^{2} y^2 \, dy = \frac{1}{3} y^3 \Big|_{x^2}^{2} = \frac{8}{3} - \frac{x^6}{3}$$

But, we still <u>can</u> switch order, but must be done carefully

$$R = \{(x,y) : 0 \le x \le \sqrt{2},\quad x^2 \le y \le 2\}$$



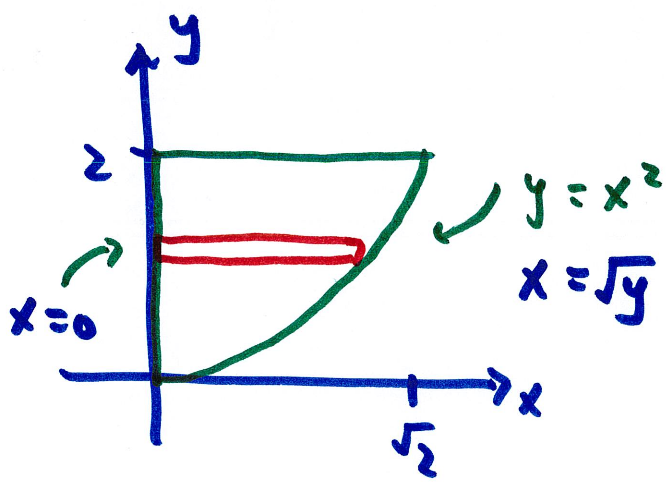
height $= 2 - x^2$

this is sometimes called a <u>Type I</u> region

bunch of vertical rectangles

x: bounded by constants

to switch order, we need to formulate R as a <u>Type II</u> region



horizontal rectangles : right curve − left curve

right : $x = \sqrt{y}$

left : $x = 0$

$$0 \le x \le \sqrt{y}$$

$$0 \le y \le 2$$

$$R: \{(x,y) : 0 \le x \le \sqrt{y},\quad 0 \le y \le 2\}$$

integrate using the new order

$$\int_0^2 \int_0^{\sqrt{y}} xy^2 \, dx \, dy = \int_0^2 \left(\frac{1}{2} x^2 y^2 \Big|_{x=0}^{x=\sqrt{y}} \right) dy$$

$y \qquad x$

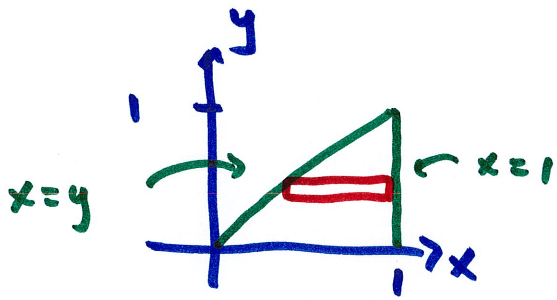
$$= \int_0^2 \frac{1}{2} y^3 \, dy = \frac{1}{8} y^4 \Big|_0^2 = \boxed{2}$$

<u>example</u>

$$\int_0^1 \int_y^1 e^{x^2}\,dx\,dy$$

R as given: $R = \{(x,y) : y \le x \le 1, \quad 0 \le y \le 1\}$ Type II

left $x=y$ right $x=1$

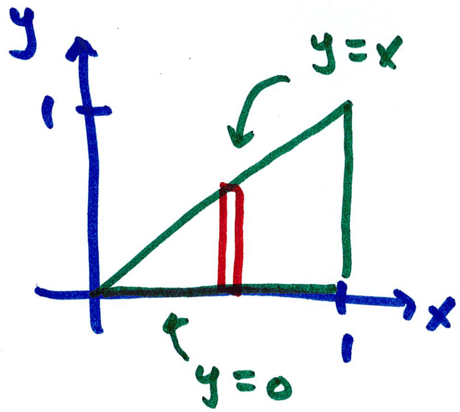


integrate using the given order:

inside $\displaystyle\int_y^1 e^{x^2}\,dx$ is not something we can ~~by~~ do by hand

we must switch to a different type of R to integrate

Switch to <u>Type I</u>



$0 \leq x \leq 1$

$0 \leq y \leq x$

bottom   top

$R = \{(x,y): \ 0 \leq x \leq 1, \ 0 \leq y \ x\}$

$$\int_0^1 \int_0^x e^{x^2} \, dy \, dx$$

inside: $\displaystyle\int_0^x e^{x^2} \, dy$

x is const

$$= e^{x^2} y \Big|_{y=0}^{y=x} = x e^{x^2}$$

$$= \int_0^1 x e^{x^2} \, dx$$

$u = x^2$

$du = 2x \, dx$

$$= \cdots = \boxed{\frac{1}{2}(e-1)}$$

Example

$$\underbrace{\int_{e^{-2}}^{1}\int_{-\ln y}^{2} f(x,y)\,dx\,dy}_{} + \underbrace{\int_{1}^{e^2}\int_{\ln y}^{2} f(x,y)\,dx\,dy}_{}$$

rewrite the integral by switching order

$e^{-2} \le y \le 1$

$-\ln y \le x \le 2$

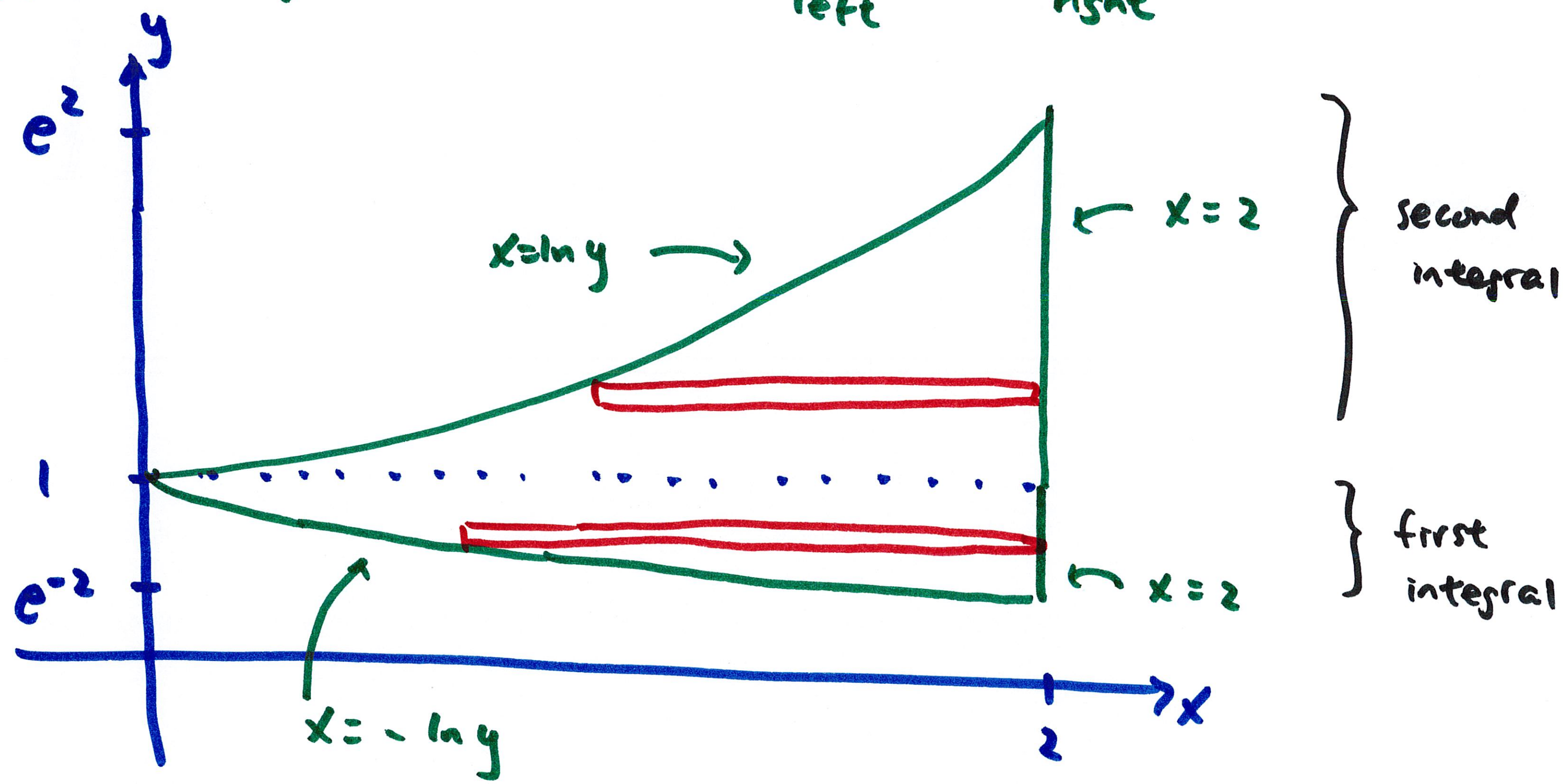
left right

$1 \le y \le e^2$

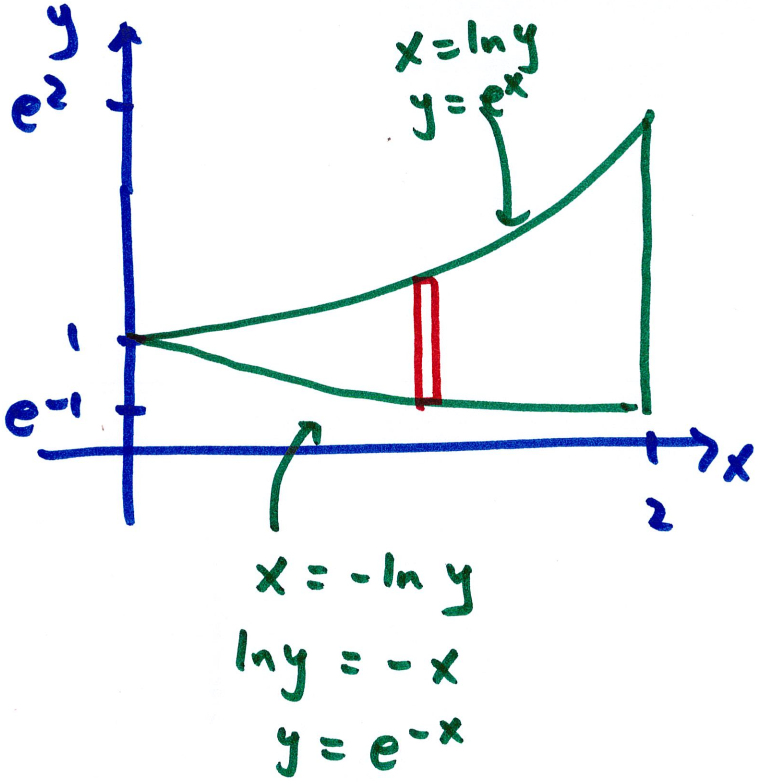
$\ln y \le x \le 2$

left right

Type II



change to <u>Type I</u> (vertical rectangles)



one integral is enough since top and bottom curves don't change

$$e^{-x} \le y \le e^x$$

$$0 \le x \le 2$$

$$R = \{(x,y): \; 0 \le x \le 2, \; e^{-x} \le y \le e^x\}$$

integral: $$\int_0^2 \int_{e^{-x}}^{e^x} f(x,y)\, dy\, dx$$