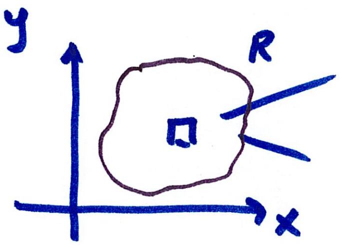
# 16.4 Triple Integrals

$\iint_R f(x,y)\,dA$ is accumulation of $f(x,y)$ over the region $R$

if $f(x,y)=1$, then $\iint_R dA$ is the area of $R$

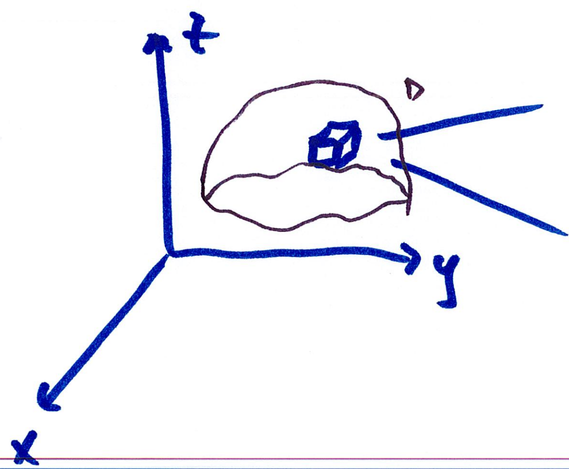


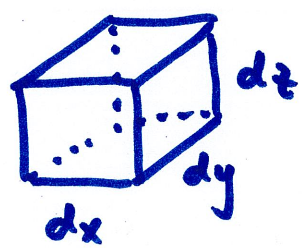


$dA = dx\,dy = dy\,dx$ two possible orders

$\iiint_D f(x,y,z)\,dV$ is the accumulation of $f(x,y,z)$ over the volume $D$

if $f(x,y,z)=1$, then $\iiint_D dV$ is the volume of $D$




$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dx\,dz\,dy \\ &= dy\,dx\,dz \\ &= dy\,dz\,dx \\ &= dz\,dx\,dy \\ &= dz\,dy\,dx \end{aligned}$$

six possible orders

<u>example</u> Use a triple integral to calculate the volume of of the solid under $\underbrace{x+2y+3z=6}_{\text{plane}}$ in the first $\underbrace{\text{octant}}_{\text{3D equivalent of quadrant}}$.

plane

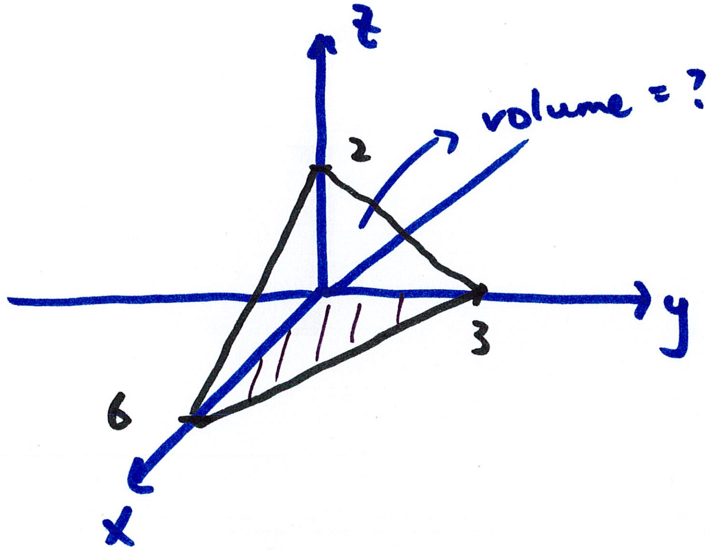
x-int : 6

y-int : 3

z-int : 2

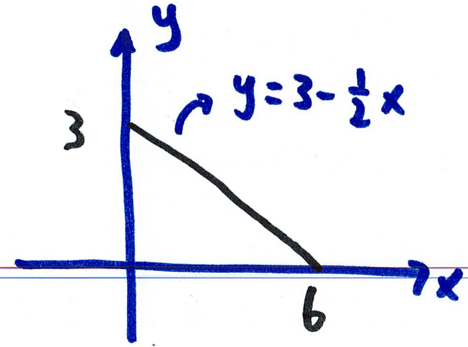
first octant

$x \geq 0,\ y \geq 0,\ z \geq 0$




$$\text{volume} = \iiint_D dV$$

let's look at the "floor" of the shape using xy-plane as a Type I region.




$0 \leq x \leq 6$

$0 \leq y \leq 3 - \frac{1}{2}x$

from plane eq.

then we will go up as high as the plane allows

$0 \leq z \leq 2 - \frac{1}{3}x - \frac{2}{3}y$

bounds: $0 \le x \le 6$

$0 \le y \le 3 - \frac{1}{2}x$

$0 \le z \le 2 - \frac{1}{3}x - \frac{2}{3}y$

Basic rule: integrate constant-bounded LAST (outside)

integrate the one with most variables FIRST (inside)

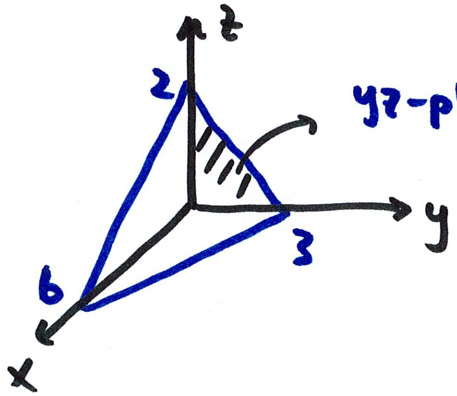
so, here

$$\int_{\underset{x}{0}}^{6} \int_{\underset{y}{0}}^{3-\frac{1}{2}x} \int_{\underset{z}{0}}^{2-\frac{1}{3}x-\frac{2}{3}y} dV \rightarrow dz\,dy\,dx$$

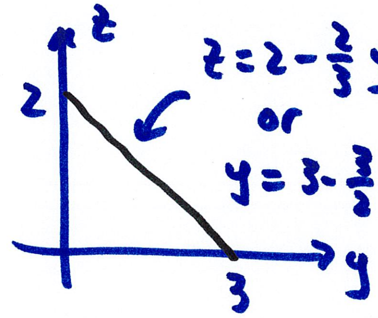
$$= \int_{0}^{6} \int_{0}^{3-\frac{1}{2}x} \int_{0}^{2-\frac{1}{3}x-\frac{2}{3}y} dz\,dy\,dx$$

$$= \int_{0}^{6} \int_{0}^{3-\frac{1}{2}x} \left(2 - \frac{1}{3}x - \frac{2}{3}y\right) dy\,dx = \cdots = \boxed{6}$$

let's try a different "floor"




yz-plane is the "floor" now




$z = 2 - \frac{2}{3}y$ or $y = 3 - \frac{3}{2}z$

Type I : $0 \le y \le 3$

$0 \le z \le 2 - \frac{2}{3}y$

Type II : $0 \le z \le 2$

$0 \le y \le 3 - \frac{3}{2}z$

we can go in x-direction as far as the plane $x + 2y + 3z = 6$ allows us

can't go below yz-plane : $x = 0$

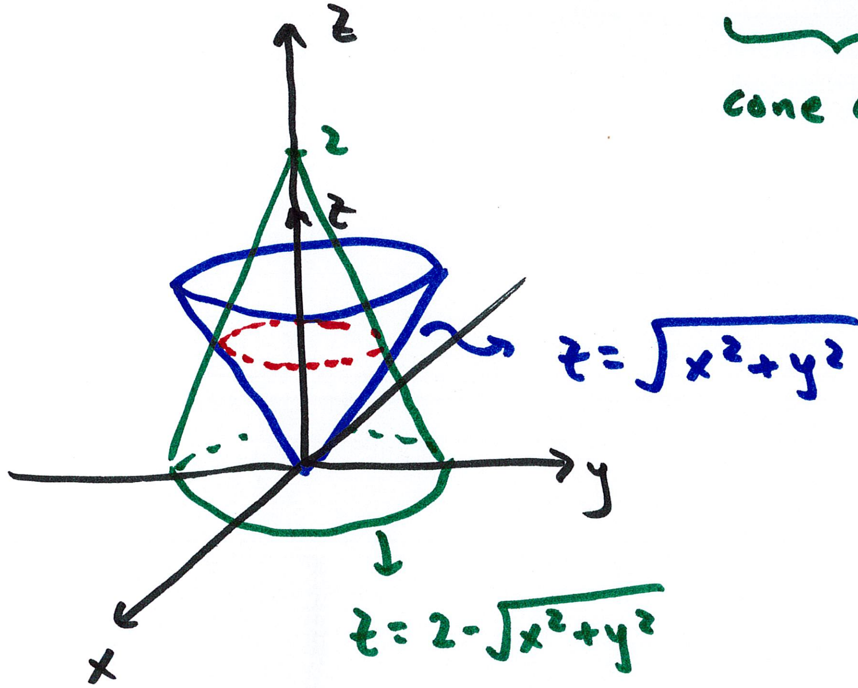
plane allows max : $x = 6 - 2y - 3z$

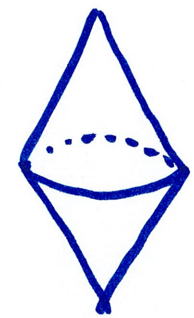
$$0 \le x \le 6 - 2y - 3z$$

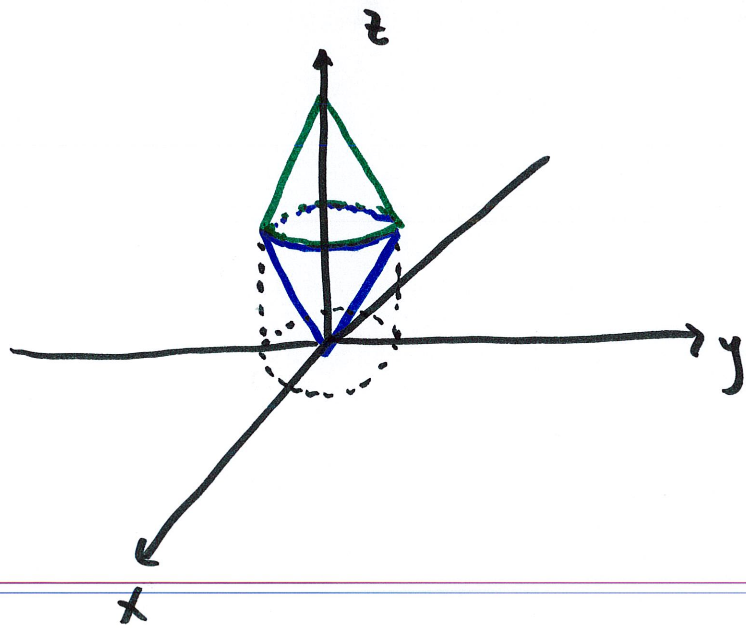
so, volume is

$$\underbrace{\int_0^2}_{z} \underbrace{\int_0^{3-\frac{3}{2}z}}_{y} \underbrace{\int_0^{6-2y-3z}}_{x} \underbrace{dx\,dy\,dz}_{dV} = \dots = \boxed{6}$$

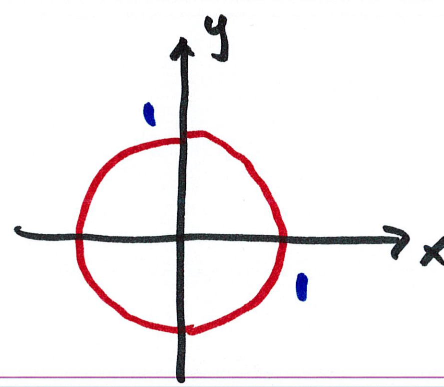
**example** Find volume above $\underbrace{z=\sqrt{x^2+y^2}}_{\text{cone opening up}}$ and below $\underbrace{z=2-\sqrt{x^2+y^2}}_{\text{cone opening down, vertex at } z=2}$




the volume of 



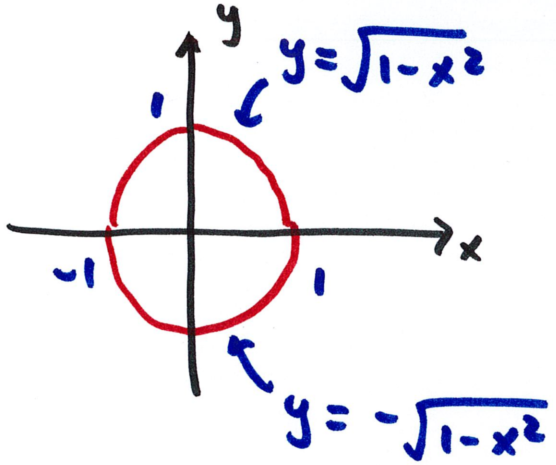

let's use the projection of this thing on the xy-plane as the "floor"




$$\sqrt{x^2+y^2} = 2-\sqrt{x^2+y^2}$$

$$\sqrt{x^2+y^2} = 1$$

$$x^2+y^2=1$$




as Type I : $-1 \le x \le 1$

$$-\sqrt{1-x^2} \le y \le \sqrt{1-x^2}$$

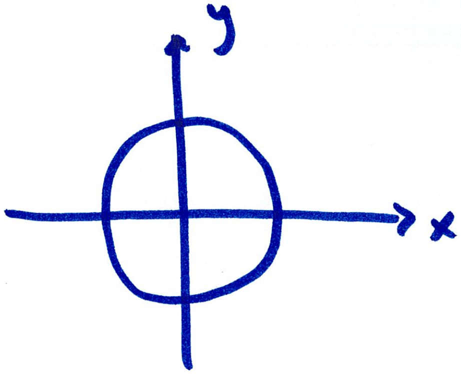
inside this region, the lowest possible $z$ is $z = \sqrt{x^2+y^2}$

" " " highest " $z$ is $z = 2-\sqrt{x^2+y^2}$

$$\sqrt{x^2+y^2} \le z \le 2-\sqrt{x^2+y^2}$$

$$\text{volume} \quad \underset{x}{\int_{-1}^{1}} \underset{y}{\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}} \underset{z}{\int_{\sqrt{x^2+y^2}}^{2-\sqrt{x^2+y^2}}} dz\,dy\,dx$$

terrible in Cartesian, let's go to polar




$0 \le r \le 1$

$0 \le \theta \le 2\pi$

"floor" was xy-plane $\rightarrow$ $dy\,dx = r\,dr\,d\theta$

$$\underbrace{\sqrt{x^2+y^2}}_{r} \le z \le \underbrace{2-\sqrt{x^2+y^2}}_{2-r}$$

$$\int_0^{2\pi}\int_0^{1}\int_r^{2-r} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{1} rz\Big|_{z=r}^{z=2-r} dr\,d\theta = \int_0^{2\pi}\int_0^{1} r(2-r-r)\,dr\,d\theta$$

$$= \dots = \boxed{\frac{2\pi}{3}}$$