16.6 Integrals for Mass Calculations

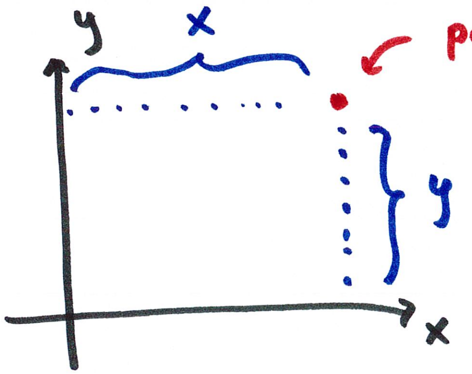
mass of shape in 2D : $\iint_R \rho(x,y)\, dA$ density

3D : $\iiint_D \rho(x,y,z)\, dA$

we will find the center of mass for 2D and 3D objects in this section

we first need the mass moment

↳ mass moment = mass · distance from the axis of rotation



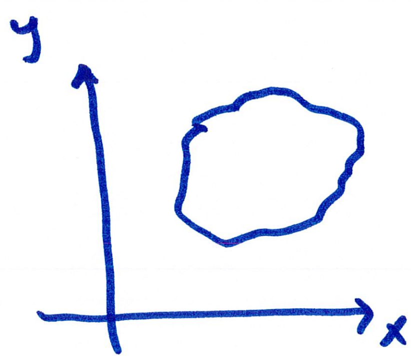
the moment of the mass about the y-axis

is M_y = mass · $\underbrace{\text{distance from y-axis}}_{\text{x-coord of point}}$

so, $\boxed{M_y = mx}$

Similarly, $\boxed{M_x = my}$

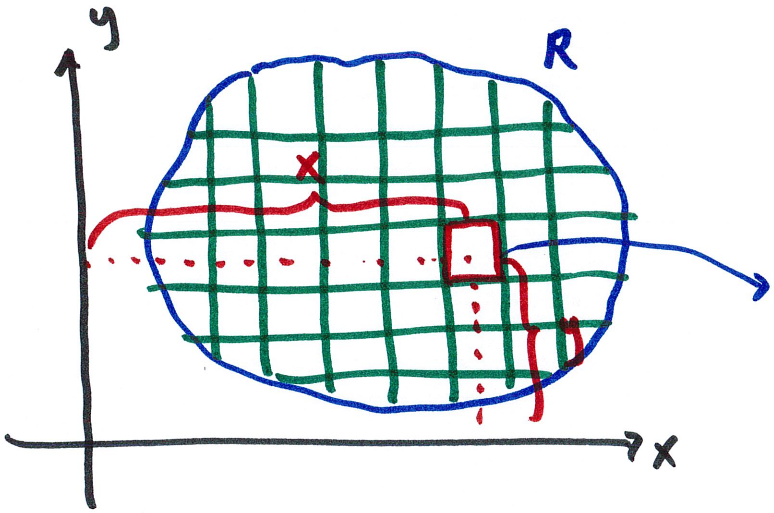
what about a 2D plate of certain shape?



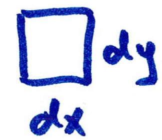
$M_y = ?$

$M_x = ?$

divide the region into small rectangles



find M_y and M_x for one rectangle, then integrate to accumulate all



mass density $\rho(x,y)$

then mass of this small piece is ~~$\iint$~~

$$\begin{aligned} m &= \rho(x,y)\,dA \\ &= \rho(x,y)\,dy\,dx \\ &= \rho(x,y)\,dx\,dy \end{aligned}$$

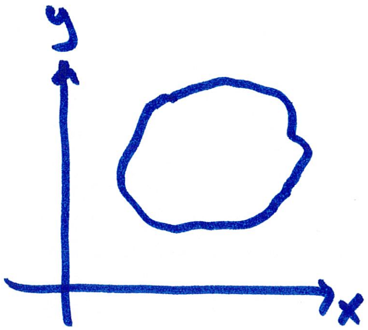
the small piece is at distance x from y-axis

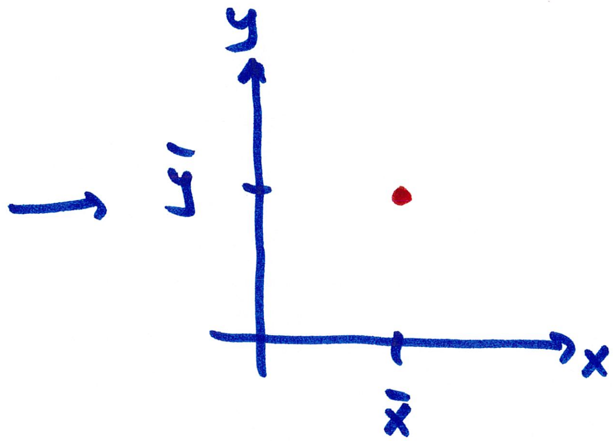
so, $M_y = \underset{\text{dist}}{x}\cdot\underbrace{\rho(x,y)\,dA}_{\text{mass}}$ and $M_x = y\cdot\rho(x,y)\,dA$

then accumulate by double integral, so, for the entire plate,

$$\boxed{M_y = \iint_R x\cdot\rho(x,y)\,dA} \qquad \boxed{M_x = \iint_R y\cdot\rho(x,y)\,dA}$$

center of mass: If the entire plate is collapsed into a point w/ same mass, where should the point be place to keep the same M_y and M_x?





location: $(\bar{x}, \bar{y})$

"x bar"

x-coord of center of mass

has

$M_y = \iint_R x\rho \, dA \longleftrightarrow$

$M_x = \iint_R y\rho \, dA \longleftrightarrow$

has

$M_y = m\bar{x}$

$M_x = m\bar{y}$

equate them: $m\bar{x} = \iint_R x\rho \, dA \longrightarrow \boxed{\bar{x} = \frac{1}{m}\iint_R x\rho \, dA}$

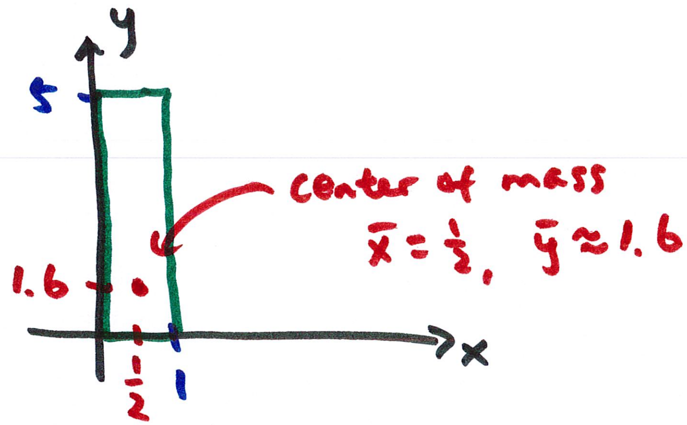
$m\bar{y} = \iint_R y\rho \, dA \longrightarrow \boxed{\bar{y} = \frac{1}{m}\iint_R y\rho \, dA}$

$\boxed{m = \iint_R \rho \, dA}$

example

$$R = \{(x,y) : 0 \le x \le 1, \quad 0 \le y \le 5\}$$

$$\rho(x,y) = 2e^{-\frac{1}{2}y}$$



start w/ mass: $m = \iint_R \rho \, dA = \int_0^1 \int_0^5 2e^{-\frac{1}{2}y} \, dy\, dx = \cdots = 4(1 - e^{-5/2})$

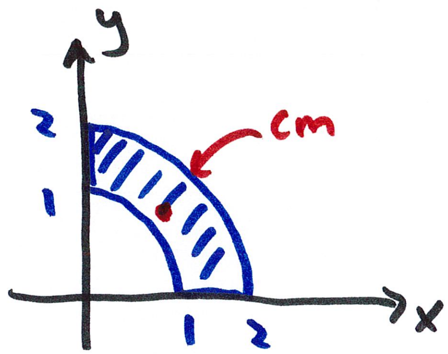
$$\bar{x} = \frac{1}{m} \iint_R x \rho \, dA = \frac{1}{4(1-e^{-5/2})} \int_0^1 \int_0^5 x \cdot 2e^{-\frac{1}{2}y} \, dy\, dx = \cdots = \boxed{\frac{1}{2}}$$

$$\bar{y} = \frac{1}{m} \iint_R y \rho \, dA = \frac{1}{4(1-e^{-5/2})} \int_0^1 \int_0^5 y \cdot 2e^{-\frac{1}{2}y} \, dy\, dx$$

int. by parts

$$= \cdots = \frac{-14e^{-5/2} + 4}{2(1-e^{-5/2})} \approx \boxed{1.6}$$

example R: ~~cen~~ between circles of radii 1 and 2 centered at origin in QI w/ density $\rho(x,y) = \sqrt{x^2+y^2}$



these are circles, so let's set up in polar

$$R = \left\{ (r,\theta) : \quad 0 \le \theta \le \tfrac{\pi}{2}, \quad 1 \le r \le 2 \right\}$$

ρ in polar: $\rho = r$

mass: $$m = \iint_R \rho \, dA = \int_0^{\frac{\pi}{2}} \int_1^2 \underset{\rho}{\underset{\downarrow}{r}} \, \underbrace{r\,dr\,d\theta}_{dA} = \int_0^{\frac{\pi}{2}} \int_1^2 r^2 \, dr\,d\theta$$

$$= \cdots = \frac{7\pi}{6}$$

$$\bar{x} = \frac{1}{m} \iint_R \overset{r\cos\theta}{x} \rho \, dA = \frac{1}{7\pi/6} \int_0^{\frac{\pi}{2}} \int_1^2 \underbrace{r\cos\theta}_{x} \cdot \underset{\rho}{\underset{\downarrow}{r}} \cdot \underbrace{r\,dr\,d\theta}_{dA} = \cdots = \frac{45}{14\pi} \approx 1.023$$

$$\bar{y} = \frac{1}{m} \iint_R y \rho \, dA = \frac{1}{7\pi/6} \int_0^{\frac{\pi}{2}} \int_1^2 r\sin\theta \cdot r \cdot r\,dr\,d\theta = \cdots = \frac{45}{14\pi} \approx 1.023$$

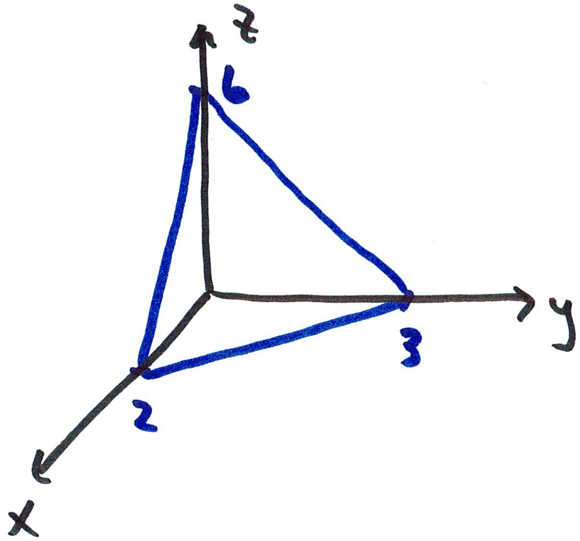
2D to 3D : $dA \to dV$ $\quad \rho(x,y) = \rho(x,y,z)$

double integrals $\to$ triple integrals

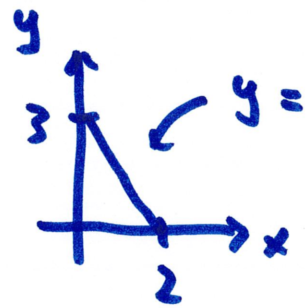
$$\boxed{\begin{aligned} m &= \iiint_D \rho(x,y,z)\,dV \\ \bar{x} &= \frac{1}{m}\iiint_D x\rho(x,y,z)\,dV \\ \bar{y} &= \frac{1}{m}\iiint_D y\rho(x,y,z)\,dV \\ \bar{z} &= \frac{1}{m}\iiint_D z\rho(x,y,z)\,dV \end{aligned}}$$

example Find center of mass of solid bounded by $\underbrace{3x+2y+z=6}_{\text{plane}}$ and the coordinate planes.

density: $\rho(x,y,z) = 1+x$



for the integrals, let's use xy-plane as the "floor"



$0 \le x \le 2$

$0 \le y \le 3-\frac{3}{2}x$

for z, from xy-plane to plane

$0 \le z \le 6-3x-2y$

mass: $$m = \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} \underbrace{(1+x)}_{\rho}\, \underbrace{dz\,dy\,dx}_{dV} = \cdots = 9$$

$$\bar{x} = \frac{1}{m} \iiint_D x \rho \, dV$$

$$= \frac{1}{9} \int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} x(1+x) \, dz \, dy \, dx = \cdots = 3/5$$

$\bar{y}, \bar{z}$ same idea ...... $\bar{y} = 7/10$, $\bar{z} = 7/5$

---

first power, so this is a <u>first moment</u>

$$M_y = \iint_R x \rho \, dA$$

$$I_y = \iint_R x^2 \rho \, dA \rightarrow \text{moment of inertia about } y\text{-axis}$$

second power

this is a <u>second moment</u>