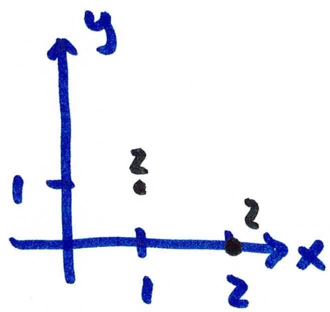
17.1 Vector Fields

(NOT on exam 2)

$f(x,y) = x+y$ this is also called a scalar field

because it assigns a scalar to each point (x,y)



at $(1,1)$ $f(1,1) = 2$

$(2,0)$ $f(2,0) = 2$

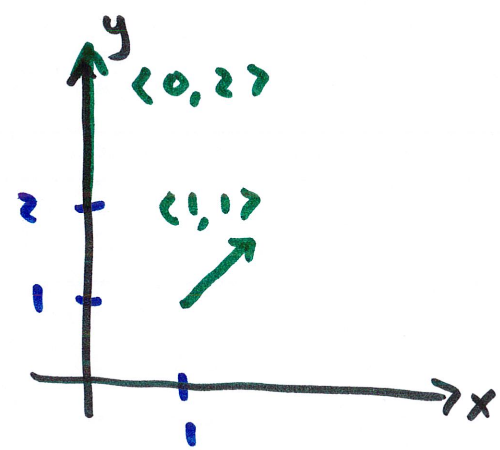
an example: temperature distribution in this room

a vector field is a function that assigns a vector to a point (x,y)

for example, $\vec{F}(x,y) = \langle x, y \rangle = x\vec{i} + y\vec{j}$

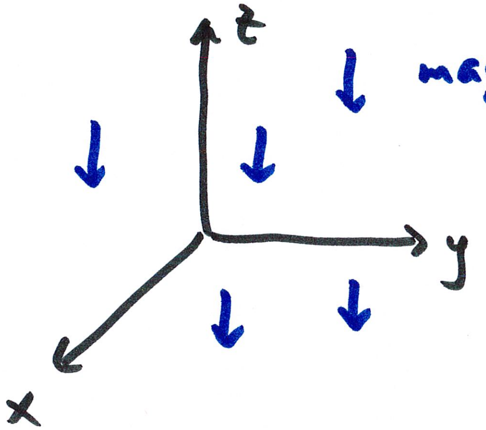
at $(1,1)$, $\vec{F}(1,1) = \langle 1, 1 \rangle = \vec{i} + \vec{j}$

$(0,2)$, $\vec{F}(0,2) = \langle 0, 2 \rangle$



the acceleration due to gravity in this room is a vector field

$$\vec{F}(x,y,z) = \langle 0, 0, -g \rangle$$



$\downarrow$ magnitude is g

<u>example</u> $$\vec{F}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

Sketch ?

we can pick points, but let's analyze this vector field a bit more

$$\vec{F} = \frac{1}{\sqrt{x^2+y^2}} \langle x, y \rangle$$

$$|\vec{F}| = \left|\frac{1}{\sqrt{x^2+y^2}}\right| \, |\langle x, y\rangle| = \frac{1}{\sqrt{x^2+y^2}}\sqrt{x^2+y^2} = 1$$

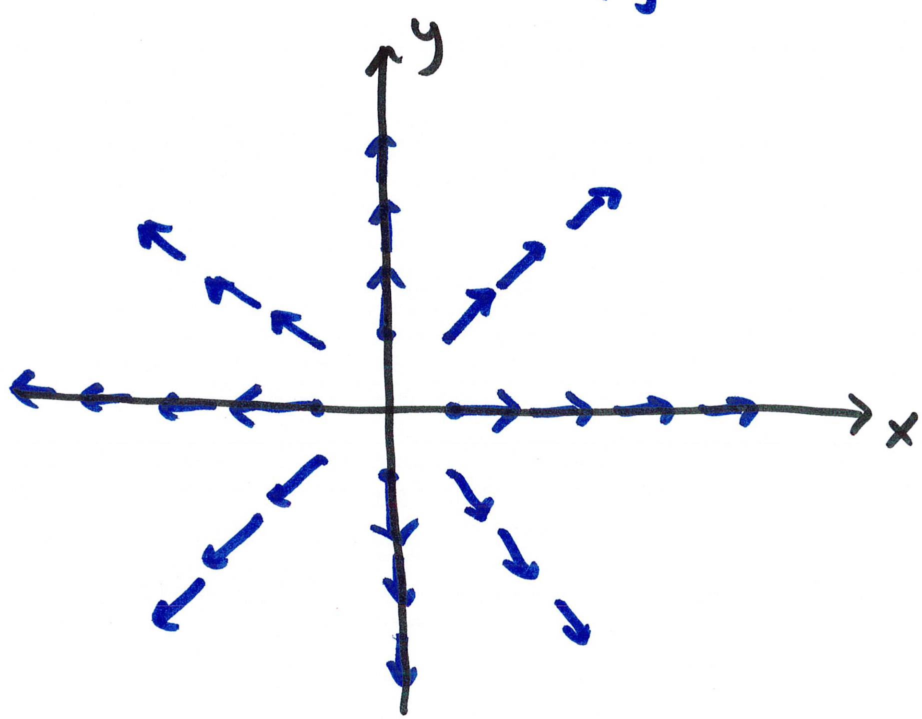
so this vector field consists of vectors of magnitude 1

direction is from $\langle x, y \rangle$ at $(1,1) \to \langle 1, 1\rangle$

$(0,1) \to \langle 0, 1 \rangle$

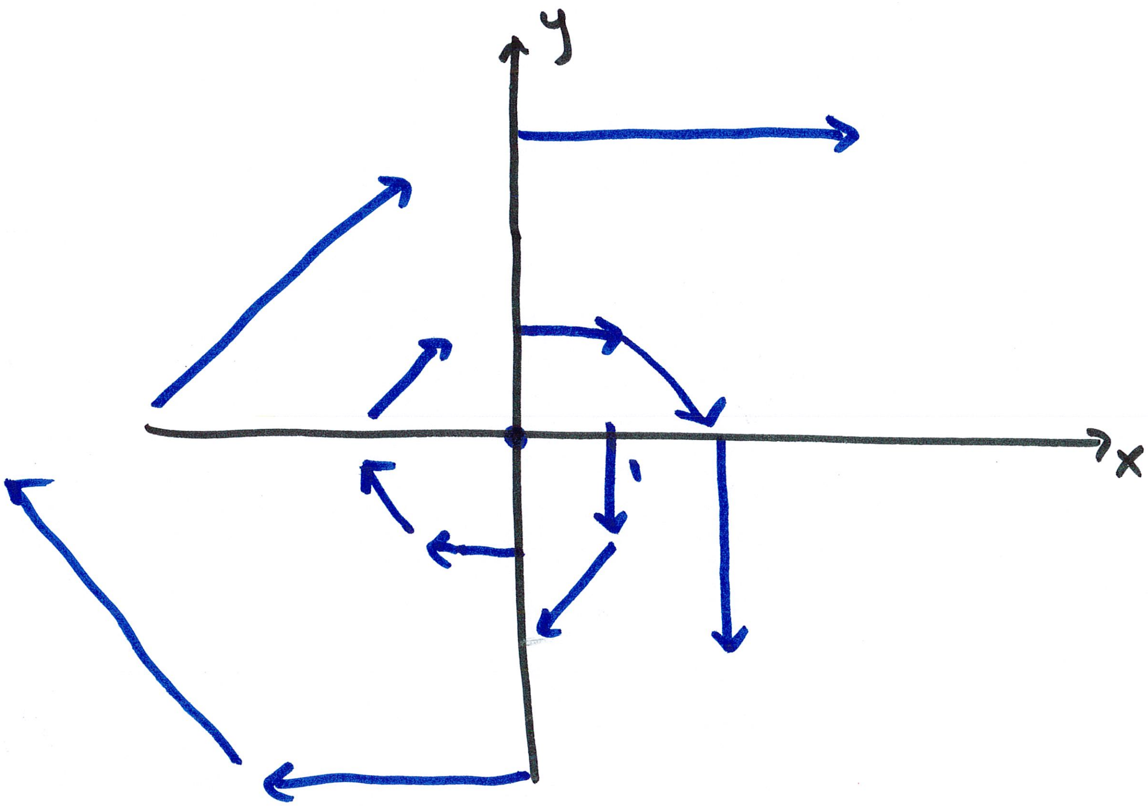
$(1,0) \to \langle 1, 0 \rangle$

$(-1,0) \to \langle -1, 0 \rangle$



unit vectors pointing radially away from origin example $\vec{F}(x,y) = \langle y, -x \rangle$

magnitude: $|\vec{F}| = \sqrt{x^2+y^2}$ → the farther from origin, the greater the magnitude



$\vec{F}(0,0) = \langle 0, 0 \rangle = \vec{0}$ (zero vector)

$\vec{F}(1,0) = \langle 0, -1 \rangle$

$\vec{F}(0,1) = \langle 1, 0 \rangle$

$\vec{F}(0,-1) = \langle -1, 0 \rangle$

$\vec{F}(1,1) = \langle 1, -1 \rangle$

$\vec{F}(2,0) = \langle 0, -2 \rangle$

Spiral out

if the vector field $\vec{F}$ is the gradient of some scalar function U, then the function U is called the potential function of the vector field $\vec{F}$.

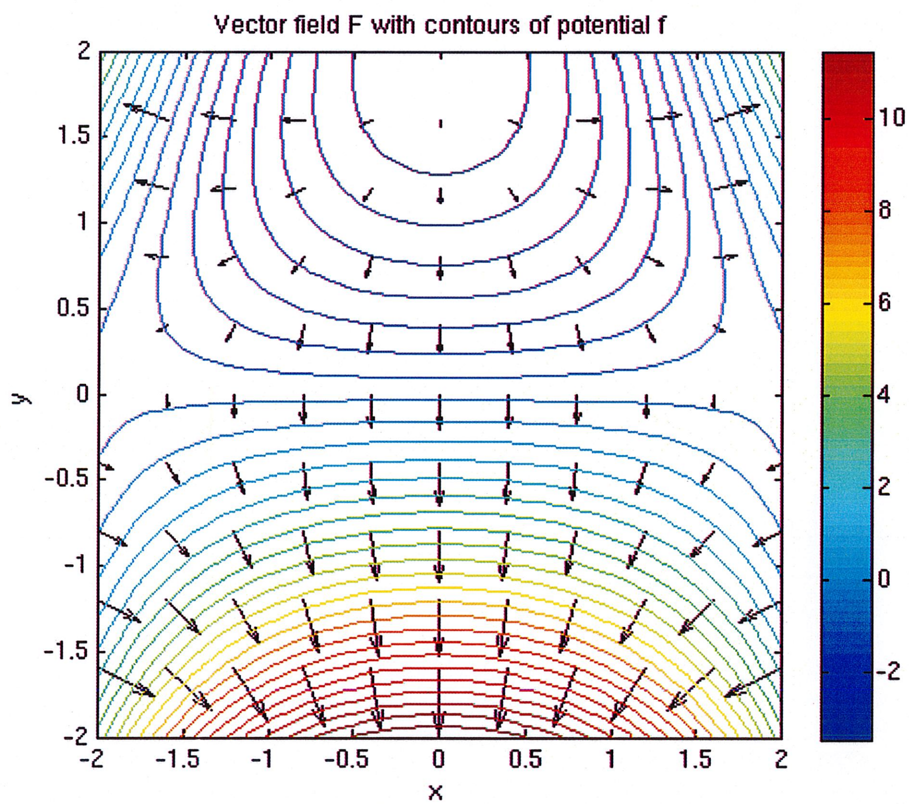
example $U = \dfrac{1}{\sqrt{x^2+y^2}}$

$$\vec{\nabla} U = \left\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y} \right\rangle = \left\langle -\frac{1}{2}(x^2+y^2)^{-3/2}(2x), -\frac{1}{2}(x^2+y^2)^{-3/2}(2y) \right\rangle$$

$$= \left\langle \frac{-x}{(\sqrt{x^2+y^2})^3}, \frac{-y}{(\sqrt{x^2+y^2})^3} \right\rangle$$

So, U is the potential function since the vector field is $\vec{\nabla} U$

if we graph them, we see the vectors of the field vector field being $\perp$ to level curves of U


Vector field F with contours of potential f
y
x
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
10
8
6
4
2
0
-2

NOT every vector field is the gradient of something

→ not every vector field has a potential function

if the vector field $\vec{F}$ represents a force vector field, and it has a potential function ($\vec{F} = \vec{\nabla} U$), then we say the force is <u>conservative</u> (gravity is one such force)

→ in a conservative force field, the work done is independent of path