17.2 (part 1) Line Integrals of Functions

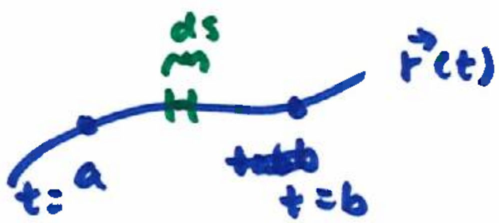
line integral: evaluation / accumulation of a function along a curve

$$\int_C f(x,y)\, ds$$

ds ← length of a small segment of curve

C: the curve

ds: we get from length:



$$\int_a^b \underbrace{\overbrace{|\vec{r}\,'(t)|\,dt}^{ds}}_{\text{length of small segment}}$$

in fact, length of curve is a special case of line integral with $f(x,y) = 1$

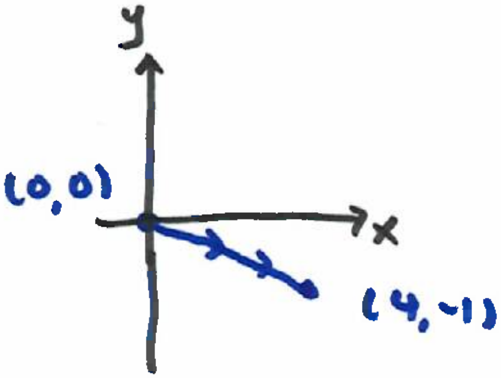
$$\int_C ds = \int_a^b |\vec{r}\,'(t)|\,dt$$

in 3D, $\int_C f(x,y,z)\, ds$

the potentially complicated part is the <u>parametrization of curve C</u>

↳ write an equation for it

<u>example</u> $\int_C x e^y \, ds$ C: line segment from $(0,0)$ to $(4,-1)$



parametrize the curve C

→ write an equation for it (line here)

$\vec{r}(t) = \vec{r}_0 + t\,\vec{v}$

$\vec{v} = \langle 4, -1 \rangle \qquad \vec{r}_0 = \langle 0, 0 \rangle$

$\vec{r}(t) = \langle 0, 0 \rangle + t \langle 4, -1 \rangle$

$$\boxed{\vec{r}(t) = \langle \underbrace{4t}_{x}, \underbrace{-t}_{y} \rangle \qquad 0 \le t \le 1} \qquad \vec{r}\,' = \langle 4, -1 \rangle$$

$ds = |\vec{r}\,'(t)|\,dt = |\langle 4, -1 \rangle|\,dt = \sqrt{4^2 + (-1)^2}\,dt = \sqrt{17}\,dt$

$$\int_C x e^y \, ds = \int_0^1 \underbrace{(4t)}_{x} e^{\overbrace{(-t)}^{y}} \underbrace{\sqrt{17}\,dt}_{ds} = 4\sqrt{17} \int_0^1 t e^{-t} \, dt = \cdots = \boxed{2\sqrt{17}\,(e-1)}$$

note $\vec{r}(t) = \langle 4t, -t \rangle$ $0 \le t \le 1$ is not the only way to parametrize C

also correct : $\vec{r}(t) = \langle 8t, -2t \rangle$ $0 \le t \le \frac{1}{2}$

or $\vec{r}(t) = \langle 2t, -\frac{1}{2}t \rangle$ $0 \le t \le 2$

does $\int_C f(x,y)\,ds$ depend on how we parametrize ?

let's check, using $\vec{r}(t) = \langle \underbrace{8t}_{x}, \underbrace{-2t}_{y} \rangle$ $0 \le t \le \frac{1}{2}$

$\vec{r}\,'(t) = \langle 8, -2 \rangle$ $ds = |\vec{r}\,'(t)|\,dt = \sqrt{68}\,dt$

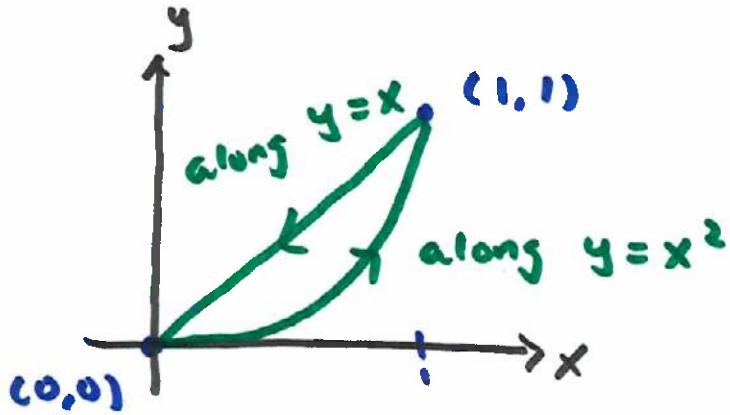
$$\int_C x e^y\,ds = \int_0^{\frac{1}{2}} 8t\,e^{-2t}\sqrt{68}\,dt$$

$$= 8\sqrt{68}\int_0^{\frac{1}{2}} t\,e^{-2t}\,dt = \dots = \text{same}$$

we don't expect it to change : if $f(x,y) = 1$, then $\int_C ds = \text{length}$
and should not be affect by choice of parametrization

example $\int_C (x+\sqrt{y})\,ds$

C : from (0,0) to (1,1) along $y=x^2$
then from (1,1) to (0,0) along $y=x$



we need to parametrize the two portions separately

C_1 : $(0,0) \to (1,1)$ along $y=x^2$

C_2 : $(1,1) \to (0,0)$ along $y=x$

C_1 : $y=x^2$, so if we let $x=t$, $0 \le t \le 1$

then $\vec{r}(t) = \langle t, t^2 \rangle$ describes C_1

or $\vec{r}(t) = \langle \sqrt{t}, t \rangle$ $\quad 0 \le t \le 1$

or $\vec{r}(t) = \langle 2t, 4t^2 \rangle$ $\quad 0 \le t \le \frac{1}{2}$

for simplicity, let's choose $\vec{r}(t) = \langle t, t^2 \rangle$ $\quad 0 \le t \le 1$
for C_1

C_2: from $(1,1)$ to $(0,0)$ along $y=x$

$\vec{r}(t) = \langle -t, -t \rangle$ $t=1$ to $t=0$ doesn't work, because we don't start at $(1,1)$

but $\vec{r}(t) = \langle t, t \rangle$ $t=1$ to $t=0$ does

how about $\vec{r}(t) = \langle 1-t, 1-t \rangle$ $0 \le t \le 1$

as long as $y=x$ (curve), and start/end are right, the parametrization should work

let's go with $\vec{r}(t) = \langle 1-t, 1-t \rangle$ $0 \le t \le 1$ for C_2

$$\int_C (x+\sqrt{y})\,ds = \int_{C_1} (x+\sqrt{y})\,ds + \int_{C_2} (x+\sqrt{y})\,ds$$

for C_1: $\vec{r}(t) = \langle t, t^2 \rangle$ $0 \le t \le 1$

$\vec{r}\,' = \langle 1, 2t \rangle$ $ds = |\vec{r}\,'|\,dt = \sqrt{1+4t^2}\,dt$

for C_2: $\vec{r}(t) = \langle 1-t, 1-t \rangle$ $0 \le t \le 1$

$\vec{r}\,' = \langle -1, -1 \rangle$ $ds = \sqrt{2}\,dt$

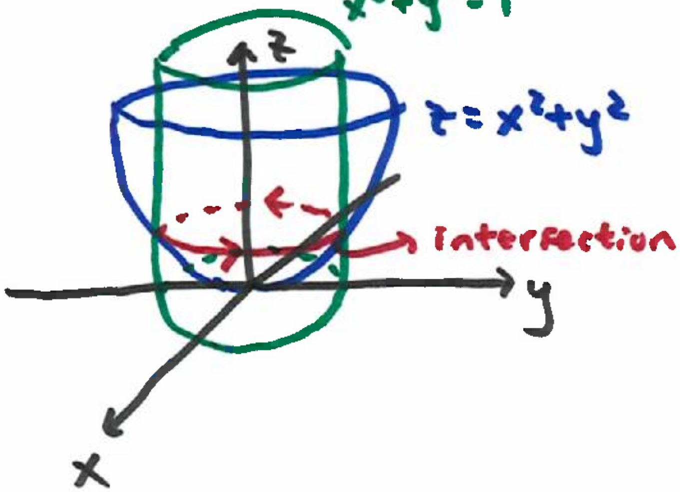
$$\underbrace{\int_0^1 \Big(\underbrace{t}_{x} + \underbrace{\sqrt{t^2}}_{y}\Big) \underbrace{\sqrt{1+4t^2}}_{ds}\,dt}_{C_1} + \underbrace{\int_0^1 \left(1-t+\sqrt{1-t}\right)\sqrt{2}\,dt}_{C_2}$$

for C_1

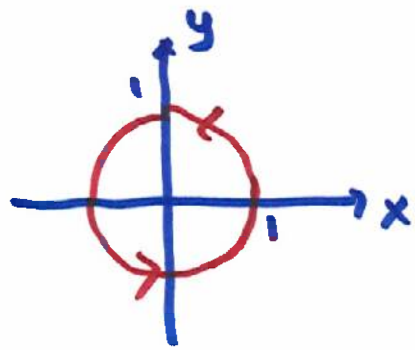
$$= \cdots = \frac{1}{6}\left(5\sqrt{5}-1\right) + \frac{7}{3\sqrt{2}}$$

example $\int_C (x+y+z)\,ds$

C : intersection of $\underbrace{z=x^2+y^2}_{\text{paraboloid}}$ and $\underbrace{x^2+y^2=1}_{\text{cylinder}}$ going counterclockwise when viewed from above



C is a circle radius 1 (part of cylinder $x^2+y^2=1$)



at $z=1$

$\downarrow$

from $z=\underbrace{x^2+y^2}_{\substack{\text{circle is radius 1} \\ \text{so } x^2+y^2=1}}$

$z=1$

now we parametrize this

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle \qquad 0 \le t \le 1$

can't have changing z-level

$\vec{r}(t) = \langle \cos t, \sin t, 1 \rangle \qquad 0 \le t \le 2\pi$

$\vec{r}(t) = \langle \cos 2\pi t, \sin 2\pi t, 1 \rangle \qquad 0 \le t \le 1$

$\vec{r}(t) = \langle \cos \pi t, \sin \pi t, 1 \rangle \qquad 0 \le t \le 2$

$x = \cos 2\pi t, \quad y = \sin 2\pi t, \quad z = 1$

$\vec{r}\,' = \langle -2\pi \sin 2\pi t, 2\pi \cos 2\pi t, 0 \rangle$

$ds = |\vec{r}\,'|\,dt = \sqrt{4\pi^2 \sin^2 2\pi t + 4\pi^2 \cos^2 2\pi t}\,dt = \sqrt{4\pi^2} = 2\pi\,dt$

$\int_C (x+y+z)\,ds = \int_0^1 (\cos 2\pi t + \sin 2\pi t + 1)(2\pi)\,dt = \cdots = \boxed{2\pi}$