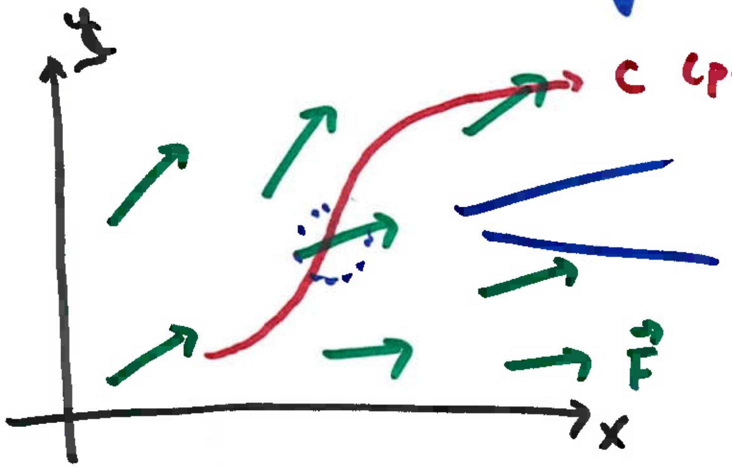
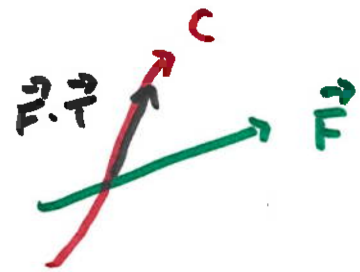
17.2 Line Integrals in a Vector Field

last time: line integral of scalar field $\int_C f(x,y)\,ds$

in a vector field, we evaluate / accumulate some component of the vector field along a path C





along the path : $\vec{F}\cdot\vec{T}$

unit tangent vector of path

$\vec{F}\cdot\vec{T}$ is scalar, so the calculation of $\int_C \vec{F}\cdot\vec{T}\,ds$ is done just like with $\int_C f(x,y)\,ds$

Look at $\boxed{\int_C \vec{F}\cdot\vec{T}\,ds}$ more closely

let's parametrize C as ~~$\vec{r}(t) = \langle$~~ $\vec{r}(t)$, $a \le t \le b$

unit tangent $\vec{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ and we know $ds = |\vec{r}\,'(t)|\,dt$

Sub into $\int_C \vec{F}\cdot\vec{T}\,ds$ we get $\int_C \vec{F}\cdot\dfrac{\vec{r}\,'(t)}{\cancel{|\vec{r}\,'(t)|}}\,\cancel{|\vec{r}\,'(t)|}\,dt = \boxed{\int_C \vec{F}\cdot\vec{r}\,'\,dt}$

another equivalent expression: $\vec{r}\,'\,dt = \dfrac{d\vec{r}}{dt}\,dt = d\vec{r}$

so, $\boxed{\int_C \vec{F}\cdot d\vec{r}}$

Common application: work in moving along C in a force vector field $\vec{F}$.

<u>example</u> $\vec{F} = \langle xy, y-x \rangle$

C: line segment from $(0,1)$ to $(2,4)$

calculate $\int_C \vec{F}\cdot\vec{T}\,ds$

first step: parametrize C

here, it's a line: $\vec{r}(t) = \vec{r}_0 + t\vec{v}$

$\vec{v} = \langle 2, 3 \rangle \qquad \vec{r}_0 = \langle 0, 1 \rangle$

$\vec{r}(t) = \langle 0, 1 \rangle + t\langle 2, 3 \rangle = \langle 2t, 1+3t \rangle \qquad 0 \le t \le 1$

remember, the parametrization of C is NOT unique

so, here, $\vec{r}(t) = \langle \underbrace{2t}_{x}, \underbrace{1+3t}_{y} \rangle \qquad 0 \le t \le 1$

$$\int_C \vec{F}\cdot\vec{T}\,ds = \underbrace{\int_C \vec{F}\cdot\vec{r}\,'\,dt}_{\text{seems easiest to use}} = \int_C \vec{F}\cdot d\vec{r}$$

$\vec{r}\,' = \langle 2, 3 \rangle$, $\quad \vec{F} = \langle xy, y-x \rangle = \langle (2t)(1+3t), 1+3t-2t \rangle$

$$\int_C \vec{F} \cdot \vec{r}\,' \, dt = \int_0^1 \langle 6t^2+2t,\ t+1 \rangle \cdot \langle 2, 3 \rangle \, dt$$

$$= \int_0^1 (12t^2 + 4t + 3t + 3) \, dt$$

$$= \cdots = \boxed{\frac{25}{2}}$$

one possible interpretation: work done in moving from $(0,1)$ to $(2,4)$ along a line in a force vector field $\vec{F} = \langle xy,\ y-x \rangle$

if C is a closed loop (start and end are the same), then

$\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{r}\,'\,dt = \int_C \vec{F}\cdot d\vec{r}$ is also called the

circulation of $\vec{F}$ on C.

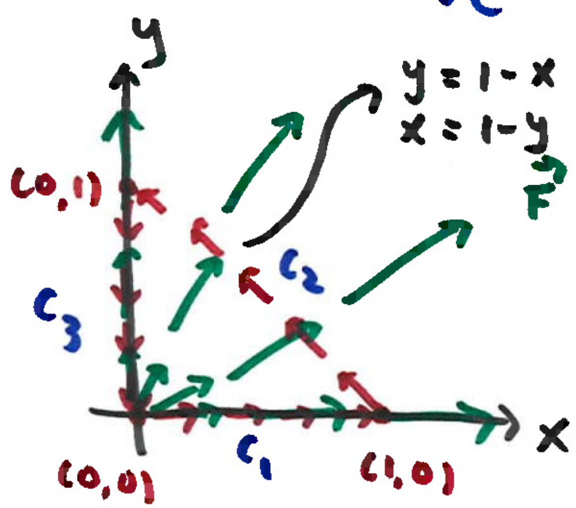
example $\vec{F} = \langle x, y \rangle$

C : $(0,0)$ to $(1,0)$ along a line segment

then $(1,0)$ to $(0,1)$ along another line segment

then $(0,1)$ to $(0,0)$ " " " "

calculate $\int_C \vec{F}\cdot\vec{T}\,ds$



we need to parametrize the segments separately

C_1 : $\vec{r}(t) = \langle t, 0 \rangle$ $\quad 0 \le t \le 1$

C_2 : $\vec{r}(t) = \langle 1-t, t \rangle$ $\quad 0 \le t \le 1$

C_3 : $\vec{r}(t) = \langle 0, 1-t \rangle$ $\quad 0 \le t \le 1$

$\int_C \vec{F}\cdot\vec{r}\,'dt$ on C_1, C_2, C_3 then add up

C_1: $\vec{F} = \langle x, y \rangle$ $\qquad \vec{r}(t) = \langle \underbrace{t}_{x}, \underbrace{0}_{y} \rangle$ $\qquad 0 \le t \le 1$

$= \langle t, 0 \rangle$ $\qquad \vec{r}\,' = \langle 1, 0 \rangle$

$$\int_{C_1} \vec{F}\cdot\vec{r}\,'dt = \int_0^1 \langle t, 0\rangle \cdot \langle 1, 0 \rangle\, dt = \int_0^1 t\, dt = \frac{1}{2}$$

C_2: $\vec{F} = \langle x, y \rangle$ $\qquad \vec{r}(t) = \langle 1-t, t \rangle$ $\qquad 0 \le t \le 1$

$= \langle 1-t, t \rangle$ $\qquad \vec{r}\,' = \langle -1, 1 \rangle$

$$\int_{C_2} \vec{F}\cdot\vec{r}\,'dt = \int_0^1 \langle 1-t, t\rangle \cdot \langle -1, 1 \rangle\, dt = \int_0^1 (2t-1)\,dt = 0$$

C_3: $\vec{F} = \langle x, y \rangle$ $\qquad \vec{r}(t) = \langle 0, 1-t \rangle$ $\qquad 0 \le t \le 1$

$= \langle 0, 1-t \rangle$ $\qquad \vec{r}\,' = \langle 0, -1 \rangle$

$$\int_{C_3} \vec{F}\cdot\vec{r}\,'dt = \int_0^1 \langle 0, 1-t\rangle \cdot \langle 0, -1 \rangle\, dt = \int_0^1 (t-1)\,dt = -\frac{1}{2}$$

$$\int_C \vec{F}\cdot\vec{r}\,'dt = \int_{C_1} \vec{F}\cdot\vec{r}\,'dt + \int_{C_2} \vec{F}\cdot\vec{r}\,'dt + \int_{C_3} \vec{F}\cdot\vec{r}\,'dt = \boxed{0}$$

another equivalent form of $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{r}\,'\,dt = \int_C \vec{F}\cdot d\vec{r}$

if $C : \vec{r}(t) = \langle x(t), y(t)\rangle \qquad a \le t \le b$

then $\frac{d\vec{r}}{dt} = \vec{r}\,' = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$

$\vec{r}\,'\,dt = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt = \langle dx, dy\rangle$

let $\vec{F} = \langle f, g\rangle$

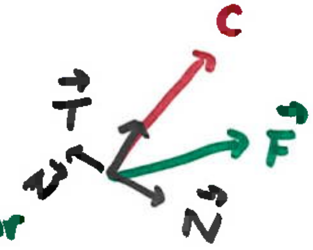
then $\vec{F}\cdot\vec{r}\,'\,dt = \langle f, g\rangle \cdot \langle dx, dy\rangle = f\,dx + g\,dy$

so, $\int_C \vec{F}\cdot\vec{T}\,ds = \boxed{\int_C f\,dx + g\,dy}$ where $\vec{F} = \langle f, g\rangle$

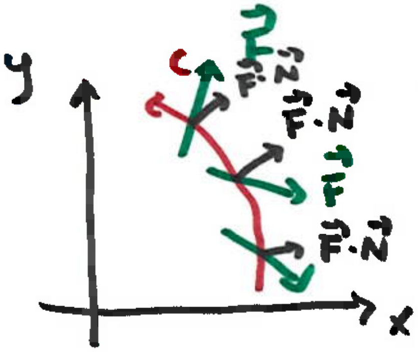
$\vec{r} = \langle x, y\rangle$

if we replace $\vec{T}$ in $\int_C \vec{F}\cdot\vec{T}\,ds$ with the component of $\vec{F}$ that is perpendicular to path instead, $\cancel{\vec{N}}$ $\vec{F}\cdot\vec{N}$

unit vector normal to C



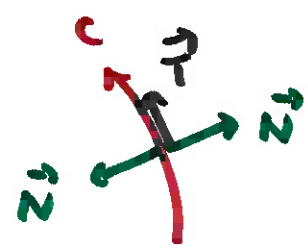
$\int_C \vec{F}\cdot\vec{T}\,ds$ becomes $\int_C \vec{F}\cdot\vec{N}\,ds$ → "flux integral"



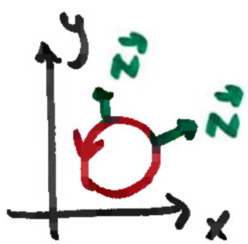
$\vec{F}\cdot\vec{N}$ is "flowing through" C

(this integral accumulates all the things flowing through the shape C)

there are two possible $\vec{N}$:



right of $\vec{T}$ or left of $\vec{T}$



Convention: if C is closed, choose $\vec{N}$ such that it points out

if C is not closed, choose $\vec{N}$ to the right of $\vec{T}$