17.3 Conservative Vector Fields and the Fundamental Theorem of Line Integrals

if $\vec{F}$ is conservative, then $\vec{F} = \vec{\nabla}\phi$ $\qquad$ ϕ: potential function

given $\vec{F}$, how do we know if it is conservative?

if so, how to find ϕ?

let $\vec{F} = \langle f, g \rangle$ be conservative

then we know $\vec{F} = \langle f, g \rangle = \vec{\nabla}\phi = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle$

so, $f = \cancel{\frac{\partial \phi}{\partial \phi}}\ \frac{\partial \phi}{\partial x}$, $\quad g = \frac{\partial \phi}{\partial y}$

furthermore, we know that mixed partials are equal: $f_{yx} = f_{xy}$

$$\underbrace{\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)}_{\phi_{xy}} = \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right)}_{\phi_{yx}} \rightarrow \boxed{f_y = g_x \text{ if } \vec{F} = \langle f, g \rangle \text{ is conservative } (\vec{F} = \vec{\nabla}\phi)}$$

example $\vec{F} = \langle x, y \rangle$

f $\quad$ g

conservative if $f_y = g_x$

$$\frac{\partial}{\partial y}(x) = \frac{\partial}{\partial x}(y) \quad ?$$

yes, $0 = 0$

so, $\vec{F} = \langle x, y \rangle$ is conservative

example $\vec{F} = \langle -y, x \rangle$

f $\quad$ g

is $\underset{f}{\frac{\partial}{\partial y}(-y)} = \underset{g}{\frac{\partial}{\partial x}(x)}$?

no, $-1 \neq 1$

so $\vec{F} = \langle -y, x \rangle$ is NOT conservative if conservative ($\vec{F} = \vec{\nabla}\phi$), how to find ϕ?

example $\vec{F} = \langle \underbrace{x+y}_{f}, \underbrace{x}_{g} \rangle$

is $f_y = g_x$? yes, so $\vec{F} = \langle x+y, x \rangle = \langle \phi_x, \phi_y \rangle = \vec{\nabla}\phi$

we see $\frac{\partial \phi}{\partial x} = x + y$ ①

$\frac{\partial \phi}{\partial y} = x$ ②

integrate ① with respect to x (treat y as constant)

① $\rightarrow$ $\phi = \int (x+y)\,dx = \frac{1}{2}x^2 + xy + \underbrace{a(y)}$

y is const.

function depends on y that vanishes when we take partial with x (to get ①) could be a constant now let's take partial with respect to y and equate with ②

$$\phi = \underbrace{\tfrac{1}{2}x^2 + xy + a(y)}$$

$$\phi_y = x + \frac{da}{dy} = x$$

$$\frac{da}{dy} = 0 \rightarrow a = c \text{ (constant)}$$

now we know ϕ: $\boxed{\phi = \tfrac{1}{2}x^2 + xy + c}$

check: is $\vec{\nabla}\phi = \vec{F} = \langle x+y,\ x \rangle$?

$$\vec{\nabla}\left(\tfrac{1}{2}x^2 + xy + c\right) = \langle x+y,\ x \rangle$$ yes.

3D: $\vec{F} = \langle f, g, h \rangle$ is it conservative ?

if conservative, $\vec{F} = \langle f, g, h \rangle = \vec{\nabla}\phi = \langle \phi_x, \phi_y, \phi_z \rangle$

$f = \phi_x$

$g = \phi_y$

$h = \phi_z$

mixed partials are equal

$\phi_{xy} = \phi_{yx}$

$\phi_{yz} = \phi_{zy}$

$\phi_{zx} = \phi_{xz}$

$$\boxed{\begin{aligned} f_y &= g_x \\ g_z &= h_y \\ h_x &= f_z \end{aligned}}$$

to recover ϕ, do similar thing as in 2D case why bother with finding ϕ?

because if $\vec{F}$ is conservative, then $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{r}\,'\,dt = \int_C \vec{F}\cdot d\vec{r}$

is <u>path-independent</u>, the only things that matter are the start and end locations.

why? if $\vec{F} = \vec{\nabla}\phi = \langle \phi_x, \phi_y \rangle$, $\vec{r}(t) = \langle x(t), y(t) \rangle$

$$\vec{F}\cdot\vec{r}\,'\,dt = \left\langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle dt$$

$$= \underbrace{\left(\frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} \right)}_{\text{result of chain rule}} dt$$

$$= \left(\frac{d\phi}{dt} \right) dt = d\phi$$

$$\int_C \vec{F}\cdot\vec{r}\,'\,dt = \int_C d\phi = \phi(B) - \phi(A)$$

end location

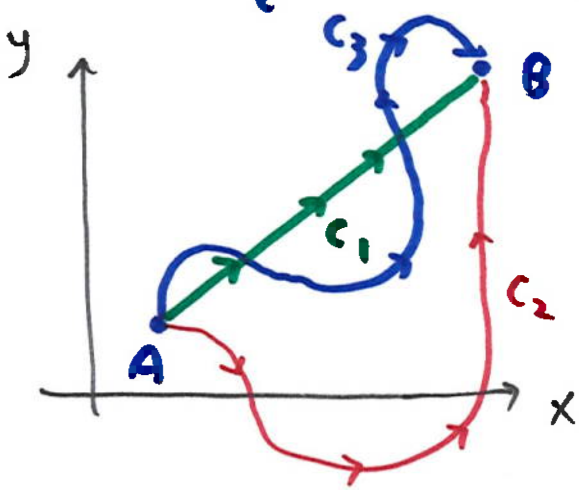
start location

Fundamental Theorem of Line Integrals

if $\vec{F}$ is conservative ($\vec{F} = \vec{\nabla}\phi$)

then $\int_C \vec{F}\cdot\vec{T}\,ds = \int_C \vec{F}\cdot\vec{r}\,'\,dt = \int_C \vec{F}\cdot d\vec{r} = \phi(B) - \phi(A)$

end ↗ ↖ start

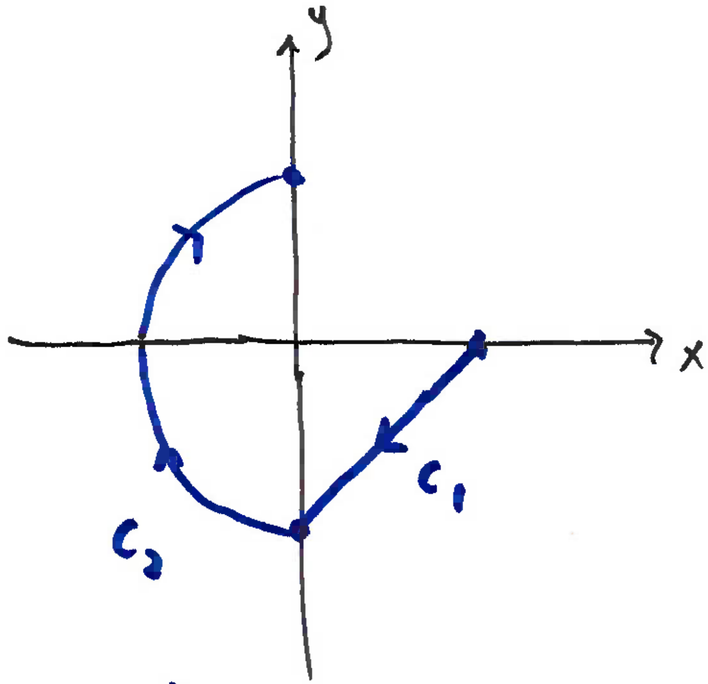


all paths lead to the same

$\int_C \vec{F}\cdot\vec{T}\,ds$ if $\vec{F} = \vec{\nabla}\phi$

<u>example</u> $\int_C \vec{F}\cdot\vec{r}\,'\,dt \qquad \vec{F} = \langle x+y,\ x\rangle$

C : line segment from $(1,0)$ to $(0,-1)$
then along left half of $x^2+y^2=1$ to $(0,1)$



let's do the "old way" first and compare to the "new way"

$C_1: \ \vec{r}(t) = \langle 1-t, -t\rangle \qquad 0\le t\le 1$

$C_2: \ \vec{r}(t) = \langle -\sin t, -\cos t\rangle \qquad 0\le t\le \pi$

$$\underbrace{\int_0^1 \langle 1-2t, 1-t\rangle\cdot\langle -1,-1\rangle\,dt}_{C_1} + \underbrace{\int_0^\pi \langle -\sin t-\cos t, -\sin t\rangle\cdot\langle -\cos t, \sin t\rangle\,dt}_{C_2}$$

$$= \int_0^1 (3t-2)\,dt + \int_0^\pi (\sin t\cos t + \cos^2 t - \sin^2 t)\,dt = \dots = -\frac{1}{2}$$

earlier we found $\vec{F} = \langle x+y, x\rangle$ to be conservative ($\vec{\nabla}\phi$)

and $\phi = \frac{1}{2}x^2 + xy + C$

let's try using the Fundamental Theorem of Line Integrals

$$\int_C \vec{F}\cdot\vec{r}\,'dt = \phi(B) - \phi(A)$$

end location $(0,1)$ start location $(1,0)$

$$= \phi(0,1) - \phi(1,0)$$

$$= \left[\frac{1}{2}(0)^2 + (0)(1) + C\right] - \left[\frac{1}{2}(1)^2 + (1)(0) + C\right]$$

$$= C - \left(\frac{1}{2} + C\right) = -\frac{1}{2}$$

alternatively, since path doesn't matter, we could replace the original path w/ simpler one but preserve start/end locations