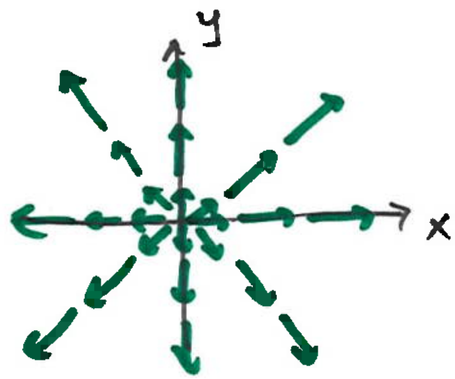
17.4 Green's Theorem

let $\vec{F} = \langle f, g \rangle$

the ~~quantity~~ vector $\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right)\vec{k} = \langle 0, 0, g_x - f_y \rangle$ is called the **curl of $\vec{F}$**, written as curl $\vec{F}$

$|\text{curl}\,\vec{F}| = |g_x - f_y|$ is a measure of rotation in $\vec{F}$

for example, $\vec{F} = \langle \underbrace{x}_{f}, \underbrace{y}_{g} \rangle$

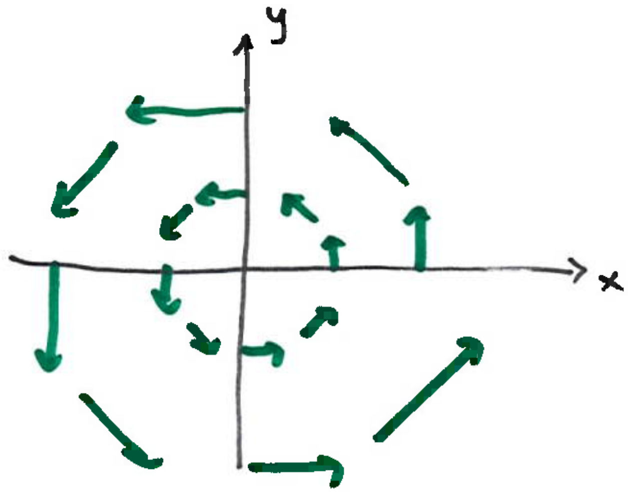


$\text{curl}\,\vec{F} = \langle 0, 0, 0-0 \rangle = \vec{0}$ $\quad |\text{curl}\,\vec{F}| = 0$

indicating no rotation

confirmed by visual inspection

$\vec{F} = \langle -y, x \rangle$



we see rotation

$\text{curl } \vec{F} = \langle 0, 0, g_x - f_y \rangle = \langle 0, 0, 1-(-1) \rangle = \langle 0, 0, 2 \rangle$

$|\text{curl } \vec{F}| = 2 \neq 0$, indicating a rotation

from last time, if $\vec{F} = \langle f, g \rangle$, then $\vec{F}$ is conservative

if $g_x = f_y \rightarrow g_x - f_y = 0$

this means if $\vec{F}$ is conservative, then $|\text{curl } \vec{F}| = 0$

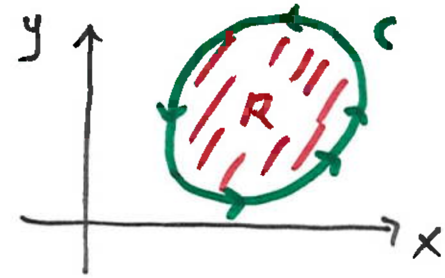
or a conservative vector field is irrotational

Green's Theorem

if $\vec{F} = \langle f, g \rangle$ and C is a simple closed path traversed once in the counterclockwise direction,

then $$\oint \vec{F} \cdot d\vec{r} = \oint f\,dx + g\,dy = \iint_R \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA$$

circle means closed path



why is this true?

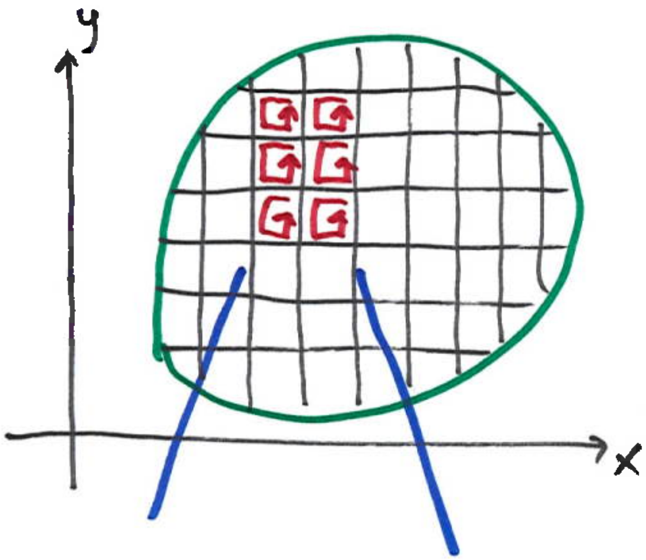
$\oint \vec{F} \cdot d\vec{r}$ is a line integral accumulating $\vec{F}$ along the path

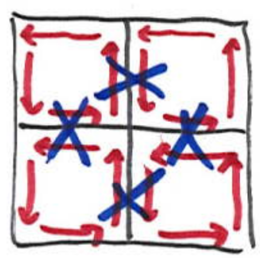
→ circulation on the boundary

right side : $\iint_R \underbrace{(g_x - f_y)}_{|\text{curl } \vec{F}|} \, dA$

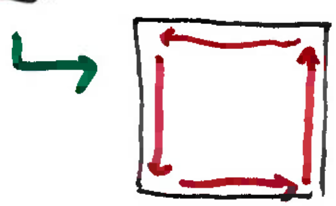
So this is accumulating rotation on region R

inside each grid there is rotation

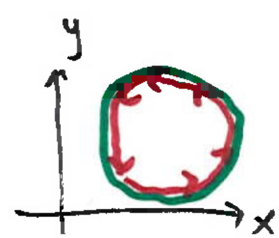




rotations cancel on boundary of two boxes



after canceling inside, ends up like

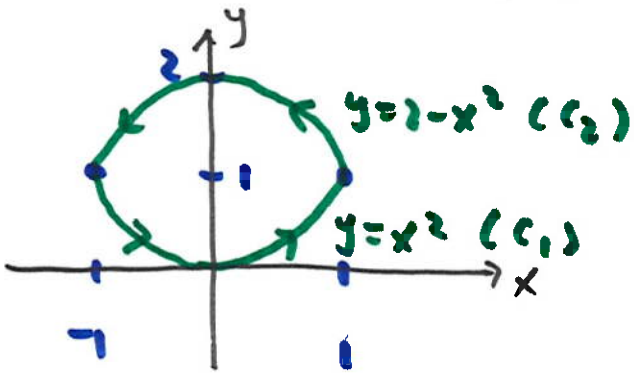


accumulates ? along boundary

→ left side of theorem

<u>example</u> $\vec{F} = \langle y+2,\ x^2+1 \rangle$

C : $(-1, 1)$ to $(1, 1)$ along $y = x^2$ then back to $(-1, 1)$ along $y = 2 - x^2$



let's do this as a line integral, then ~~we~~ try the Green's Theorem way

C_1 : $\vec{r}(t) = \langle t, t^2 \rangle \qquad -1 \le t \le 1$

C_2 : $\vec{r}(t) = \langle -t, 2-t^2 \rangle \qquad -1 \le t \le 1$

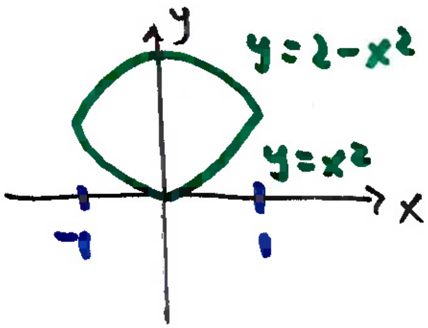
$$\int_C \vec{F} \cdot \vec{r}\,' dt = \underbrace{\int_{-1}^{1} \langle t^2+2,\ t^2+1 \rangle \cdot \langle 1,\ 2t \rangle dt}_{C_1}$$

$$+ \underbrace{\int_{-1}^{1} \langle 4-t^2,\ t^2+1 \rangle \cdot \langle -1,\ -2t \rangle dt}_{C_2}$$

$$= \int_{-1}^{1} (2t^3+t^2+2t+2)\,dt + \int_{-1}^{1} (-2t^3+t^2-2t-4)\,dt = \cdots = \frac{14}{3} - \frac{22}{3} = \boxed{-\frac{8}{3}}$$

Green's Theorem says we can trade line integral for $\iint_R (g_x - f_y)\,dA$

$\vec{F} = \langle y+2,\ x^2+1 \rangle$ $\quad g_x - f_y = 2x - 1$



R : $\quad -1 \le x \le 1$

$\quad x^2 \le y \le 2 - x^2$

$$\iint_R (g_x - f_y)\,dA = \int_{-1}^{1} \int_{x^2}^{2-x^2} (2x-1)\,dy\,dx$$

$$= \int_{-1}^{1} (2x-1)(2-2x^2)\,dx = \int_{-1}^{1} (-4x^3+2x^2+4x-2)\,dx$$

$$= \cdots = \boxed{-\frac{8}{3}}$$

Green's Theorem can also find area of R as a line integral

$$\oint f\,dx + g\,dy = \iint_R (g_x - f_y)\,dA$$

if $g_x - f_y = 1$, then right side is $\iint_R dA$ = area of R

now come up with a vector field

$\vec{F} = \langle f, g \rangle$ such that $g_x - f_y = 1$

one possibility: $\vec{F} = \langle \underbrace{-\frac{1}{2}y}_{f}, \underbrace{\frac{1}{2}x}_{g} \rangle$

use these on left side

$$\oint f\,dx + g\,dy = \oint -\frac{1}{2}y\,dx + \frac{1}{2}x\,dy$$

$$= \boxed{\frac{1}{2}\oint x\,dy - y\,dx}$$ this gives area of R

Green's Theorem $\oint f\,dx + g\,dy = \iint_R (g_x - f_y)\,dA$ is based on

$\int \vec{F}\cdot\vec{T}\,ds$ on the left side

↳ unit tangent

if we use the unit normal $\vec{N}$ instead of unit tangent $\vec{T}$

we end up with $\oint f\,dy - g\,dx = \iint_R \underbrace{(f_x + g_y)}\,dA$

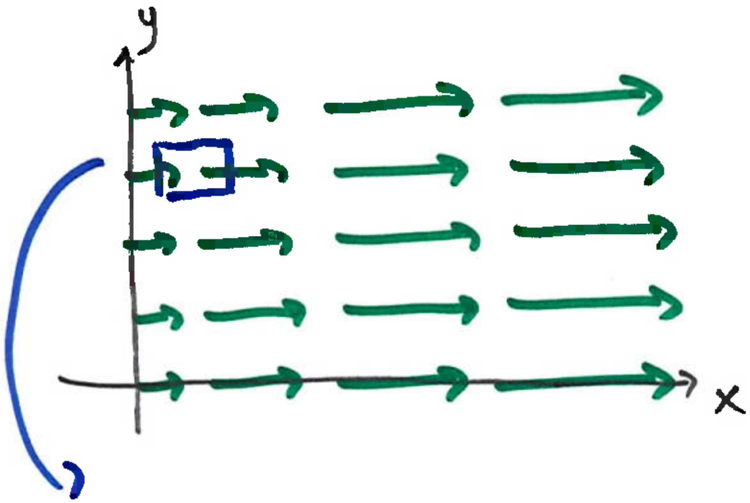
this is called the Divergence of $\vec{F} = \text{div}\,\vec{F}$

(in "standard" Green's Theorem, right side is curl $\vec{F}$)

the divergence measures ~~of m~~ the change of a small volume as it travels in the vector field, and in this alternate Green's Theorem form we get the <u>flux</u>

illustration of divergence

$\vec{F} = \langle x, 0 \rangle$ $\quad$ $\text{div}\,\vec{F} = f_x + g_y = 1 + 0 = 1$ → this means the rate of increase in volume of a small box as it moves in $\vec{F}$ is positive



Small ~~magnitde~~ magnitude on left than right so it ends being stretched, so volume increases (positive divergence)