17.6 Surface Integrals (part 2)

last time: $\displaystyle\iint_S g(x,y,z)\,dS = \underbrace{\iint_S g(u,v)\,|\vec{r}_u \times \vec{r}_v|\,dA}_{\text{general formula, works in any coordinate system}}$ ← dA: dudv or dvdu

often we use Cartesian: $u=x$, $v=y$. the resulting formula is useful to know

surface S: $\vec{r}(u,v) = \vec{r}(x,y) = \langle x, y, \overbrace{f(x,y)}^{z} \rangle$

$$\left.\begin{aligned} \vec{r}_u &= \vec{r}_x = \langle 1, 0, f_x \rangle \\ \vec{r}_v &= \vec{r}_y = \langle 0, 1, f_y \rangle \end{aligned}\right\} \quad \begin{aligned} \vec{r}_u \times \vec{r}_v &= \langle -f_x, -f_y, 1 \rangle \\ |\vec{r}_u \times \vec{r}_v| &= \sqrt{1 + f_x^2 + f_y^2} \end{aligned}$$

$$dS = \sqrt{1+f_x^2+f_y^2}\,\overbrace{dy\,dx}^{\text{or } dx\,dy}$$

→ area of a small patch of S

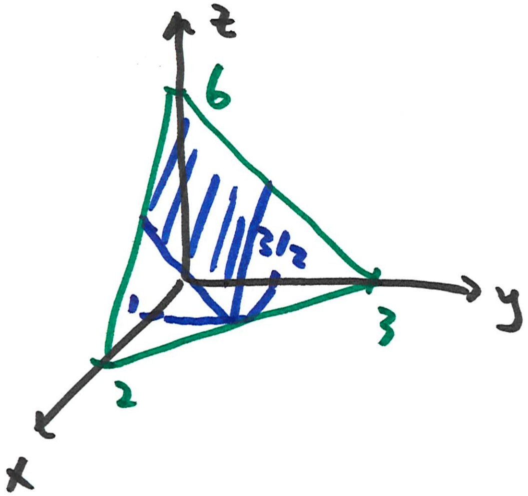
notice similarity to length of curve $L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ if $y=f(x)$

so, in Cartesian, the surface integral can be written as

$$\boxed{\iint_R g(x,y,z)\sqrt{1+f_x^2+f_y^2}\,dA}$$

<u>example</u> $\iint\limits_S (x+y)\,dS$ $\qquad$ S: $z = 6-3x-2y$ above the rectangle

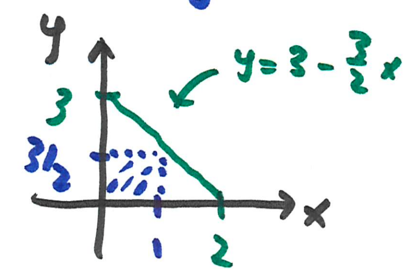
$0 \le x \le 1, \quad 0 \le y \le 3/2$



Surface : $z = f(x,y) = 6-3x-2y$

$f_x = -3 \quad f_y = -2$

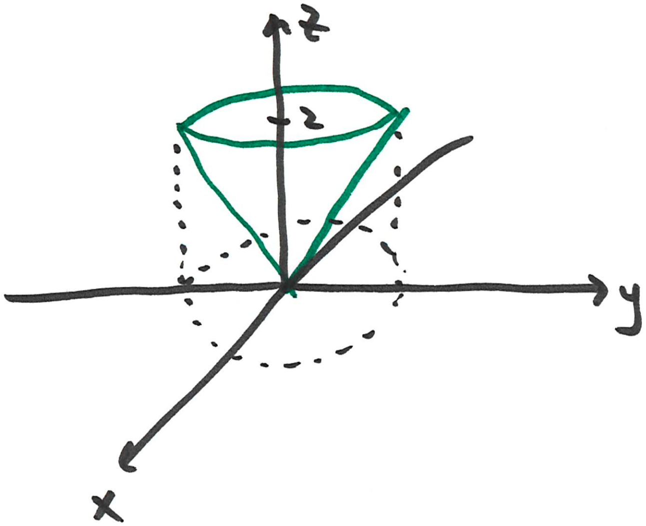
$dS = \sqrt{1+f_x^2+f_y^2}\,dA = \sqrt{14}\,dy\,dx$



$$\iint\limits_S (x+y)\,dS = \int_0^1 \int_0^{3/2} (x+y)\sqrt{14}\,dy\,dx$$

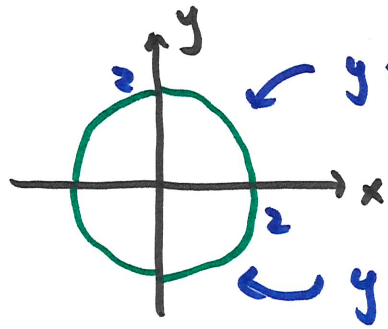
$$= \cdots = \frac{15\sqrt{14}}{8}$$

example Find the surface area of $z^2 = x^2 + y^2$ $\quad 0 \le z \le 2$



$z = f(x,y) = \sqrt{x^2+y^2}$

projection onto xy-plane



$y = \sqrt{4-x^2}$

$y = -\sqrt{4-x^2}$

$-2 \le x \le 2 \qquad -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

$dS = \sqrt{1+f_x^2+f_y^2}\, dA \qquad f_x = \dfrac{x}{\sqrt{x^2+y^2}} \qquad f_y = \dfrac{y}{\sqrt{x^2+y^2}}$

$$= \sqrt{1 + \frac{x^2}{(\sqrt{x^2+y^2})^2} + \frac{y^2}{(\sqrt{x^2+y^2})^2}} = \sqrt{\frac{2(x^2+y^2)}{x^2+y^2}} = \sqrt{2}$$

$dS = \sqrt{2}\, dy\, dx$

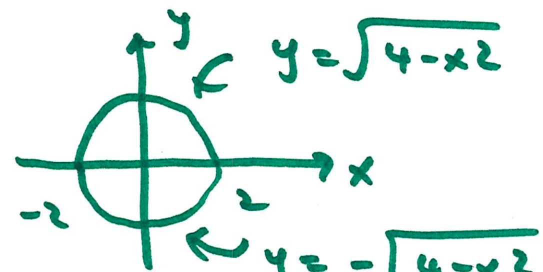
surface area: $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{2}\, dy\, dx$

$$= \sqrt{2}\underbrace{\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\, dx}$$

geometric interpretation:

area of circle radius 2

$= \pi(2)^2 = 4\pi$



$$= \sqrt{2}\cdot 4\pi = \boxed{4\sqrt{2}\,\pi}$$

alternative: change to polar

$$\sqrt{2}\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\, dx = \sqrt{2}\int_{0}^{2\pi}\int_{0}^{2} r\, dr\, d\theta$$

$\theta \quad r$

because $dA = r\,dr\,d\theta$ in polar

$$= \boxed{4\sqrt{2}\,\pi}$$

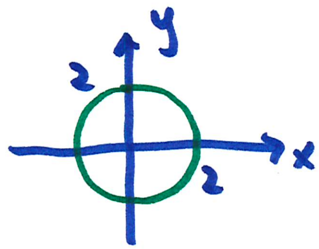
$dS = \sqrt{1+f_x^2+f_y^2}\,dA$ is assumed to be in Cartesian

if we change to anything else, we need to manually adjust dA

but $dS = |\vec{r}_u \times \vec{r}_v|\,dA$ ALWAYS accounts for dA automatically

<u>example</u> (same cone) $z^2 = x^2+y^2$ $\rightarrow$ $z = \sqrt{x^2+y^2}$ $\rightarrow$ $z = \sqrt{r^2} = r$ $\quad 0 \le z \le 2$

now let's parametrize in cylindrical



$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$\downarrow$

$0 \le u \le 2$

$0 \le v \le 2\pi$

$u = r, \quad v = \theta$

$x = r\cos\theta = u\cos v$

$y = r\sin\theta = u\sin v$

$z = z = r = u$

$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ u \rangle$

$$\left.\begin{aligned} \vec{r}_u &= \langle \cos v,\ \sin v,\ 1 \rangle \\ \vec{r}_v &= \langle -u\sin v,\ u\cos v,\ 0 \rangle \end{aligned}\right\} \quad \begin{aligned} \vec{r}_u \times \vec{r}_v &= \langle -u\cos v,\ -u\sin v,\ u \rangle \\ |\vec{r}_u \times \vec{r}_v| &= \sqrt{2}\,\overset{r}{u} \end{aligned}$$

$$dS = |\vec{r}_u \times \vec{r}_v|\,dA = \sqrt{2}\,\overbrace{u\ du\,dv}^{r\,dr\,d\theta}$$

notice $|\vec{r}_u \times \vec{r}_v|$ ALWAYS provides the "missing" piece in dA

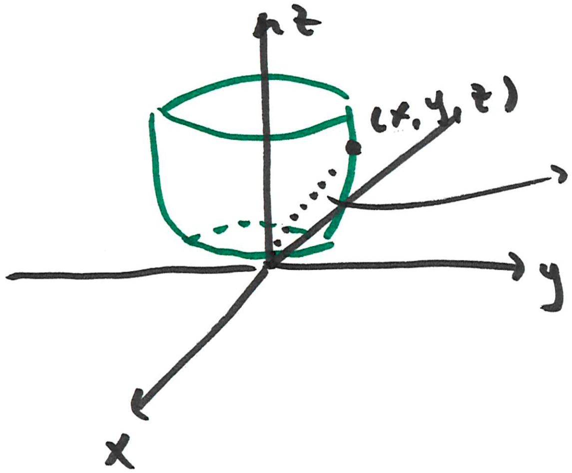
we can calculate the average value of $f(x,y,z)$ on surface S

using $$f_{avg} = \frac{\iint_S f \, dS}{\iint_S dS}$$

physically meaning: $\iint_S f \, dS$ = mass if f is density

$f_{avg} \underbrace{\iint_S dS}_{\text{surface area}}$ = mass with constant density f_{avg} over the same surface

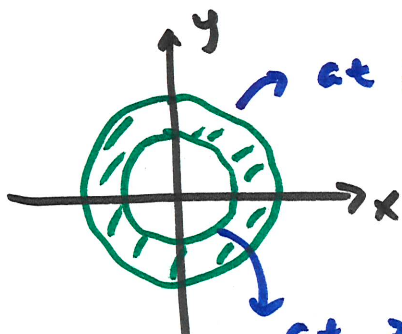
example Find the average distance of points on $z = x^2 + y^2$, $1 \le z \le 4$ from the origin



$d = \sqrt{x^2+y^2+z^2}$ distance from $(0,0,0)$

this is the function to average

parametrize. coord. sys? cylindrical is good.



at $z = 4 \rightarrow x^2+y^2 = 4$

$r^2 = 4 \rightarrow r = 2$

at $z = 1 \rightarrow r = 1$

$1 \le r \le 2$	$u = r$	$1 \le u \le 2$
$0 \le \theta \le 2\pi$	$v = \theta$	$0 \le v \le 2\pi$

$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ u^2 \rangle$ $z = x^2+y^2 = r^2 = u^2$

$$\vec{r_u} = \langle \cos v, \sin v, 2u \rangle$$

$$\vec{r_v} = \langle -u\sin v, u\cos v, 0 \rangle$$

$$|\vec{r_u} \times \vec{r_v}| = \cdots = u\sqrt{1+4u^2}$$

$$f_{avg} = \frac{\iint_S f\,dS}{\iint_S dS}$$

$$d = \sqrt{x^2+y^2+z^2}$$

$$\iint_S f\,dS = \int_{\underset{v}{0}}^{2\pi} \int_{\underset{u}{1}}^{2} \sqrt{u^2+u^4} \cdot \underbrace{u\sqrt{1+4u^2}\,du\,dv}_{dS} = \cdots = 2\pi\,(15.21)$$

$$\iint_S dS = \int_0^{2\pi}\int_1^2 u\sqrt{1+4u^2}\,du\,dv = \cdots = 2\pi\,(4.91)$$

$$f_{avg} = \frac{2\pi\,(15.21)}{2\pi\,(4.91)} \approx 3.1$$