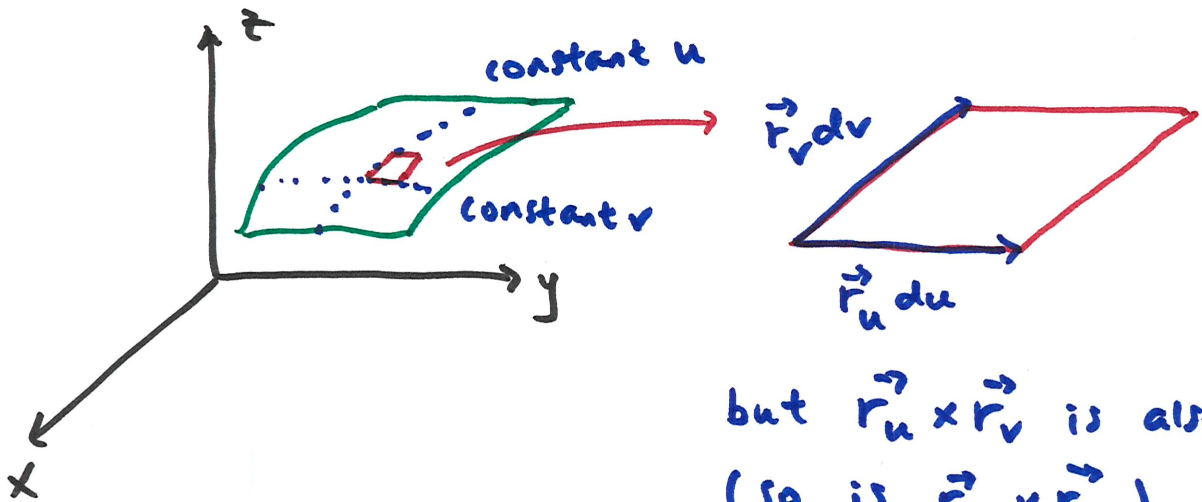
17.6 Surface Integrals (part 3)

last two times: surface integrals in <u>scalar fields</u>

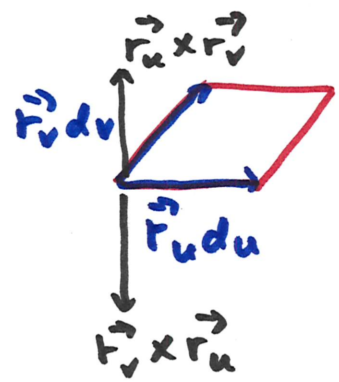
$$\iint_S f(x,y,z)\,dS = \iint_R f(u,v)\,\underbrace{|\vec{r}_u \times \vec{r}_v|\,dA}_{\text{area of small patch of surface } S}$$



area $= |\vec{r}_u \times \vec{r}_v|\,du\,dv$

but $\vec{r}_u \times \vec{r}_v$ is also the normal vector

(so is $\vec{r}_v \times \vec{r}_u$)



in scalar field, we work with magnitude of normal, so order of cross product doesn't matter.

in vector field, direction matters

so, which one to choose?

by convention, we choose the normal vector that is pointing **upward** or **outward** unless otherwise specified.

a surface with the "correct" normal is said to be **positively-oriented**

Surface integral in vector field : $\iint_S \vec{F} \cdot d\vec{S}$

vector field

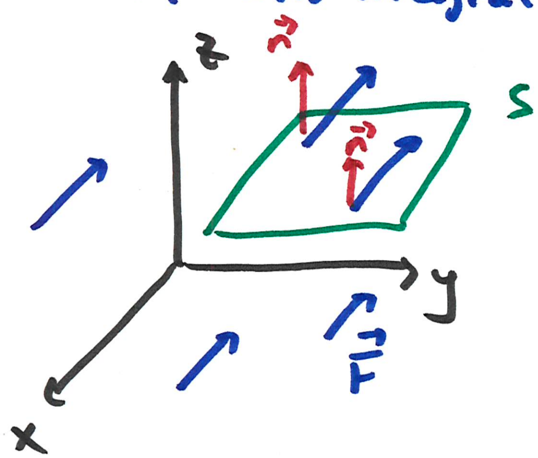
~~denotes~~ denote an oriented surface

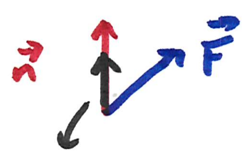
$= \iint_S \vec{F} \cdot \vec{n}\, dS$

same dS as in scalar field version

unit normal in the correct direction

this integral accumulates the component of $\vec{F}$ that is in the same direction as $\vec{n}$. This integral is also called the **Flux integral**





$\vec{F} \cdot \vec{n}$ (the component perpendicularly thru surface) how to calculate $\displaystyle\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS$?

unit normal $\displaystyle \vec{n} = \frac{\vec{r_u}\times\vec{r_v}}{|\vec{r_u}\times\vec{r_v}|}$ or $\displaystyle \frac{\vec{r_v}\times\vec{r_u}}{|\vec{r_v}\times\vec{r_u}|}$ depending on which is in the positive direction

$dS = |\vec{r_u}\times\vec{r_v}|\, du\, dv$ or $|\vec{r_v}\times\vec{r_v}|\, dA$

then $\displaystyle\iint_S \vec{F}\cdot\vec{n}\, dS$ becomes

$$\iint_S \vec{F}\cdot \frac{\vec{r_u}\times\vec{r_v}}{\cancel{|\vec{r_u}\times\vec{r_v}|}}\, \cancel{|\vec{r_u}\times\vec{r_v}|}\, dA = \boxed{\iint_R \vec{F}\cdot \underbrace{(\vec{r_u}\times\vec{r_v})}_{\text{or } \vec{r_v}\times\vec{r_u} \text{ depending on which one is positively oriented}}\, dA}$$

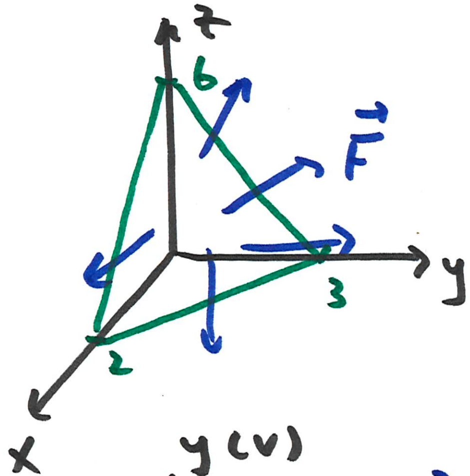
alternative (explicit) form if $z = z(x,y)$

$\vec{F} = \langle f, g, h \rangle$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_R (-f z_x - g z_y + h)\, dA$$

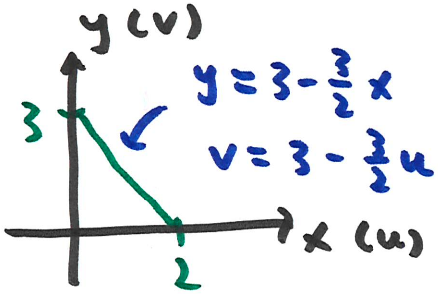
example $\vec{F} = \langle x, y, z \rangle$ S: plane $3x+2y+z=6$ in the first octant

the normal vector is considered positive upward



first, parametrize surface

let $u=x$, $v=y$. $z = 6-3x-2y = 6-3u-2v$



$0 \le u \le 2$

$0 \le v \le 3-\frac{3}{2}u$

$\vec{r}(u,v) = \langle u, v, 6-3u-2v \rangle$

$\vec{r}_u = \langle 1, 0, -3 \rangle$

$\vec{r}_v = \langle 0, 1, -2 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle 3, 2, 1 \rangle$

Stop. Is this pointing in the correct direction

is this pointing upward?

yes, b/c z-component > 0

so, use $\langle 3, 2, 1 \rangle$ as ~~a~~ correct normal

$$\iint_S \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\, dA \qquad \vec{F} = \langle x, y, z\rangle \qquad \vec{r}(u,v) = \langle \underbrace{u}_{x}, \underbrace{v}_{y}, \underbrace{6-3u-2v}_{z}\rangle$$

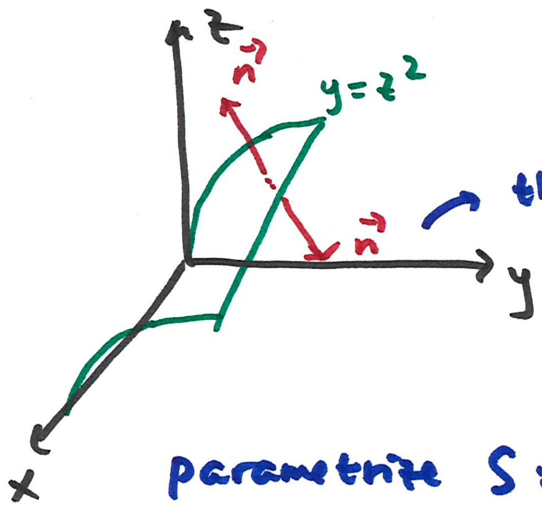
$$= \int_{u=0}^{2} \int_{v=0}^{3-\frac{3}{2}u} \underbrace{\langle u, v, 6-3u-2v\rangle}_{\vec{F}} \cdot \underbrace{\langle 3, 2, 1\rangle}_{\vec{r}_u \times \vec{r}_v}\, dv\, du$$

$$= \cdots = \boxed{18}$$

<u>example</u> $\vec{F} = \langle -y, x, 1 \rangle$

S: cylinder $y = z^2$, $\quad 0 \le x \le 3, \quad 0 \le z \le 1$

normal is considered positive pointing toward positive y-axis



this is the one with component in pos. y-axis direction
(we want this one)

parametrize S: $y = z^2$

we are given bounds in x and z
so it's convenient to use them as
parameters

let $u = x$, $\quad v = z$ $\quad$ then $y = z^2 = v^2$

$$\vec{r}(u,v) = \langle \underset{x}{u}, \underset{y}{v^2}, \underset{z}{v} \rangle$$

$0 \le u \le 3$

$0 \le v \le 1$

$\vec{r_u} = \langle 1, 0, 0 \rangle$

$\vec{r_v} = \langle 0, 2v, 1 \rangle$

$\vec{r_u} \times \vec{r_v} = \langle 0, -1, 2v \rangle$

is this normal pointing in the "positive" direction?

no, the y-component is negative so this points in negative y-axis direction

fix: $\vec{r_v} \times \vec{r_u} = \langle 0, 1, -2v \rangle$

$$\iint_S \vec{F} \cdot (\vec{r_v} \times \vec{r_u})\, dA = \int_0^3 \int_0^1 \langle -v^2, u, 1 \rangle \cdot \langle 0, 1, -2v \rangle \, dv\, du$$

$$= \cdots = \boxed{\frac{3}{2}}$$

<u>example</u> $\vec{F} = \dfrac{-\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}}$

S: sphere radius a

normal positive pointing <u>outward</u>

parametrize S: spherical $\rho = a$, $u = \phi$, $v = \theta$

$0 \leq u \leq \pi \qquad 0 \leq v \leq 2\pi$

$$\vec{r}(u,v) = \langle a\sin u\cos v,\ a\sin u\sin v,\ a\cos u\rangle$$

$\rho\sin\phi\cos\theta$

$$\vec{r}_u = \langle a\cos u\cos v,\ a\cos u\sin v,\ -a\sin u\rangle$$

$$\vec{r}_v = \langle -a\sin u\sin v,\ a\sin u\cos v,\ 0\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle a^2\sin^2 u\cos v,\ a^2\sin^2 u\sin v,\ a^2\cos u\sin u\rangle$$

"out" → all components positive in first octant

here, all correct

$$\vec{F} = \frac{-\langle x, y, z\rangle}{(x^2+y^2+z^2)^{3/2}} = \frac{-\langle a\sin u\cos v,\ a\sin u\sin v,\ a\cos u\rangle}{a^3}$$

$$\iint_S \vec{F}\cdot\underbrace{(\vec{r}_u\times\vec{r}_v)}_{\text{see last page}}\,dA = \cdots = \boxed{-4\pi}$$