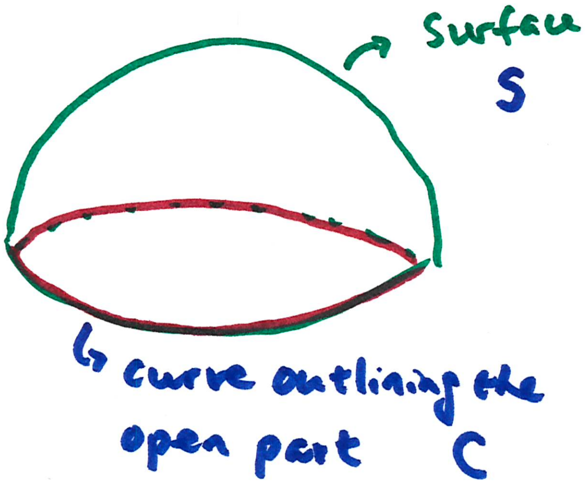
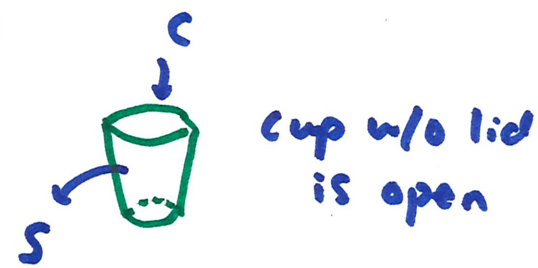
17.7 Stokes' Theorem (part 2)

Stokes' Theorem allows us to trade the surface integral of the curl of vector field for a line integral on the boundary of the surface.

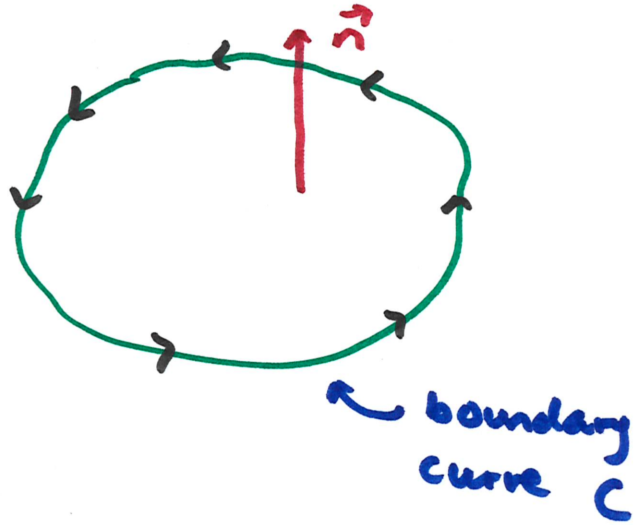


the surface must be open for Stokes' Theorem (has a hole)

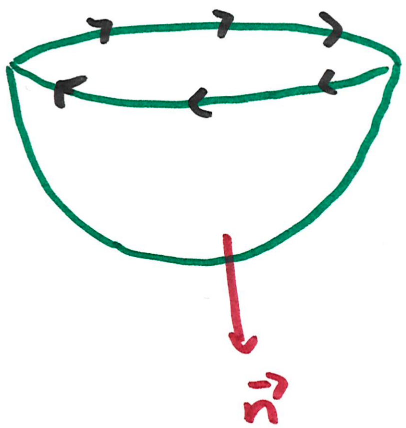


both the surface and the boundary curve are oriented
they obey the right-hand rule

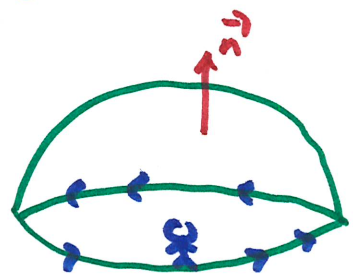
Surface oriented by the normal vector



align thumb of right hand w/ $\vec{n}$
the other fingers naturally curl
in the direction that is the
positive orientation for the curve



another way to orient both:
imagine walking along the boundary
w/ head pointing toward positive normal,
then the enclosed region is on your
left



Stokes' Theorem

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

S: surface

C: boundary curve

$$d\vec{S} = \vec{n}\, dS$$

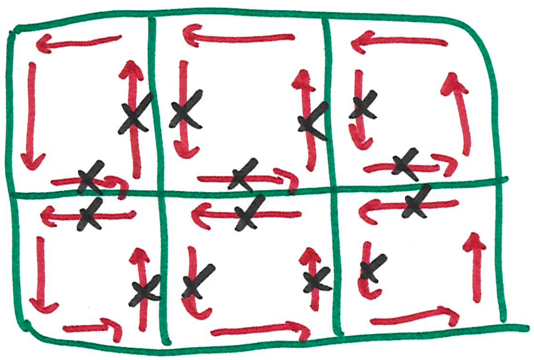
$$= (\vec{r}_u \times \vec{r}_v)\, dA \text{ or } (\vec{r}_v \times \vec{r}_u)\, dA$$

whichever is the positive normal

(convention: upward or outward unless ~~otherwi~~ otherwise stated)

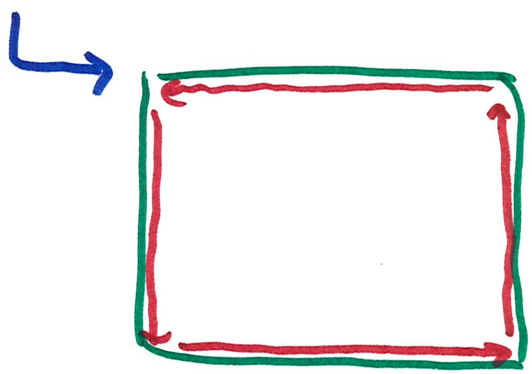
why is $\underbrace{\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}}_{\text{accumulation of curl of vector field on surface}} = \underbrace{\oint_C \vec{F} \cdot d\vec{r}}_{\text{circulation on boundary}}$

left side: $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$



in each grid we accumulate curl

along common edge → cancellation of flow



just the flow along the boundary

→ line integral $\oint_C \vec{F}\cdot d\vec{r}$

Stokes' works just like how Green's works

in fact, Green's is a special case of Stokes'

Stokes' → works for curved surface (e.g. hemisphere)

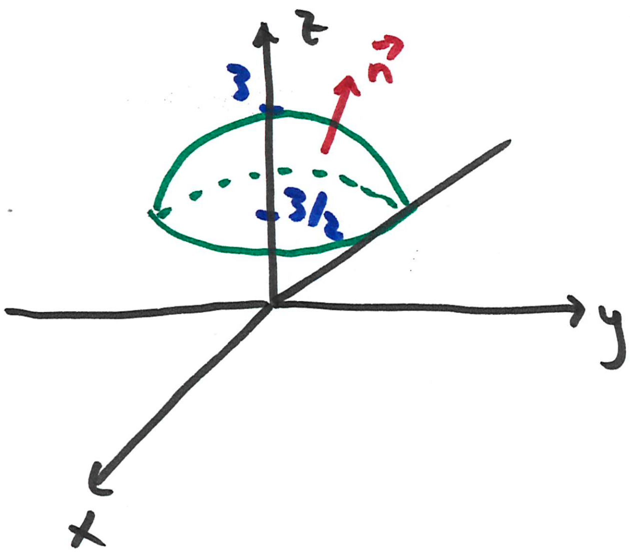
Greens' → flat surface only

<u>example</u> $\vec{F} = \langle y, -x, 0 \rangle$

S: part of sphere radius 3, $z \geq 3/2$

normal is positive pointing outward

verify Stokes' Theorem: $\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$



left side.

parametrize S first

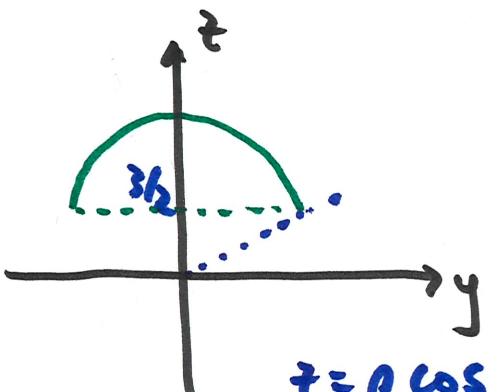
let: use spherical $\rho = 3$, so use $u = \phi$, $v = \theta$

$x = \rho \sin\phi \cos\theta \;=\; 3 \sin u \cos v$

$y = \rho \sin\phi \sin\theta \;=\; 3 \sin u \sin v$

$z = \rho \cos\phi \;=\; 3 \cos u$

$0 \leq u \leq \frac{\pi}{3} \qquad 0 \leq v \leq 2\pi$



$z = \rho \cos\phi = 3 \cos\phi$

$\frac{3}{2} = 3\cos\phi \rightarrow \cos\phi = \frac{1}{2} \rightarrow \phi = \frac{\pi}{3}$

$\vec{r}(u,v) = \langle 3\sin u\cos v,\ 3\sin u\sin v,\ 3\cos u\rangle$

$\vec{r}_u = \langle 3\cos u\cos v,\ 3\cos u\sin v,\ -3\sin u\rangle$

$\vec{r}_v = \langle -3\sin u\sin v,\ 3\sin u\cos v,\ 0\rangle$

$\vec{r}_u \times \vec{r}_v = \langle 9\sin^2 u\cos v,\ 9\sin^2 u\sin v,\ 9\sin u\cos u\rangle$ is this "outward"?

if so, $z > 0$

$0 \le u \le \frac{\pi}{3}$ $\rightarrow$ $\cos u, \sin u > 0$

$0 \le v \le 2\pi$ so $z > 0$

so, yes.

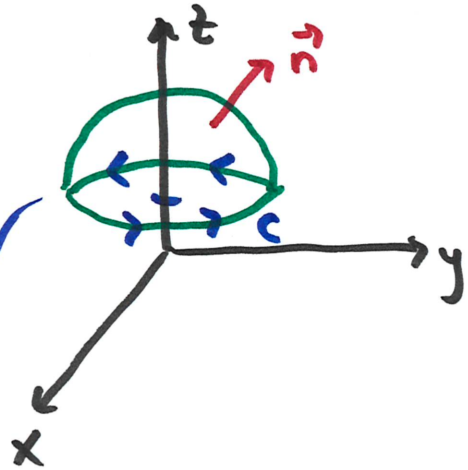
$\vec{F} = \langle y, -x, 0\rangle$ curl $\vec{F} = \vec{\nabla}\times\vec{F} = \cdots = \langle 0, 0, -2\rangle$

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_R \operatorname{curl}\vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,dA$$

$$= \int_0^{2\pi}\int_0^{\pi/3} \langle 0, 0, -2\rangle\cdot\langle \ldots, \ldots, 9\sin u\cos u\rangle\,du\,dv$$

$$= \int_0^{2\pi}\int_0^{\pi/3} -18\sin u\cos u\,du\,dv = \cdots = \boxed{-27\pi/2}$$

Stokes' says we can do $\oint_C \vec{F}\cdot d\vec{r}$ instead



boundary is counterclockwise if viewed from above because $\vec{n}$ is out / up

C : circle radius $\frac{\sqrt{27}}{2}$

$$\vec{r}(t) = \left\langle \frac{\sqrt{27}}{2}\cos t, \frac{\sqrt{27}}{2}\sin t, \frac{3}{2} \right\rangle$$

$$0 \le t \le 2\pi$$

$$x^2+y^2+z^2 = 3^2$$

at $z = 3/2$

$$x^2+y^2 = 9 - \frac{9}{4} = \frac{27}{4}$$

$$\vec{F} = \langle y, -x, 0\rangle = \left\langle \frac{\sqrt{27}}{2}\sin t, -\frac{\sqrt{27}}{2}\cos t, 0 \right\rangle$$

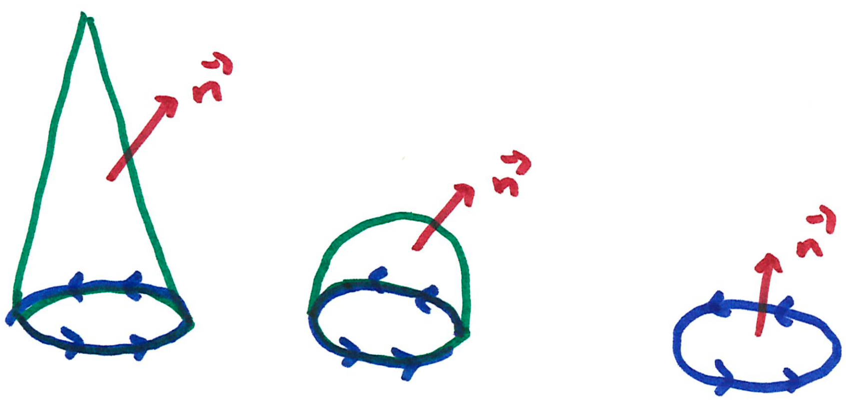
$$d\vec{r} = \vec{r}\,'dt = \left\langle -\frac{\sqrt{27}}{2}\sin t, \frac{\sqrt{27}}{2}\cos t, 0 \right\rangle$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\left(-\frac{27}{4}\sin^2 t - \frac{27}{4}\cos^2 t\right)dt = -\frac{27}{4}\int_0^{2\pi} dt$$

$$= -\frac{27}{4}\cdot 2\pi = \boxed{-\frac{27}{2}\pi}$$

Stokes' Theorem : $\displaystyle\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$

if different surfaces have the same boundary curve C,
then $\displaystyle\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$ will be the same for all



same boundary

all these surfaces have the same $\displaystyle\iint_S \text{curl}\,\vec{F}\cdot d\vec{S}$

another option: trade for a simpler surface w/ the same C
in previous example, we could have used a flat circle surface at $z = 3/2$