17.8 The Divergence Theorem (part 1)

divergence : $\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$

$$\operatorname{div}\vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle f, g, h \rangle = f_x + g_y + h_z$$

$\iiint_E \operatorname{div}\vec{F}\, dV$ accumulates the divergence inside an enclosed volume

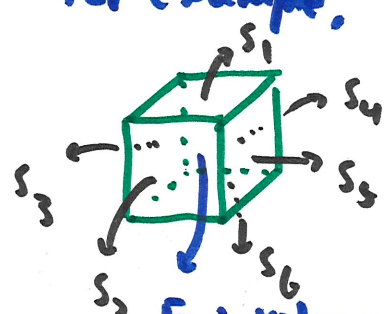
Divergence Theorem

$$\iiint_E \operatorname{div}\vec{F}\, dV = \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS$$

E : enclosed volume

S : the surface bounding the volume (closed surface)

for example, for a cube



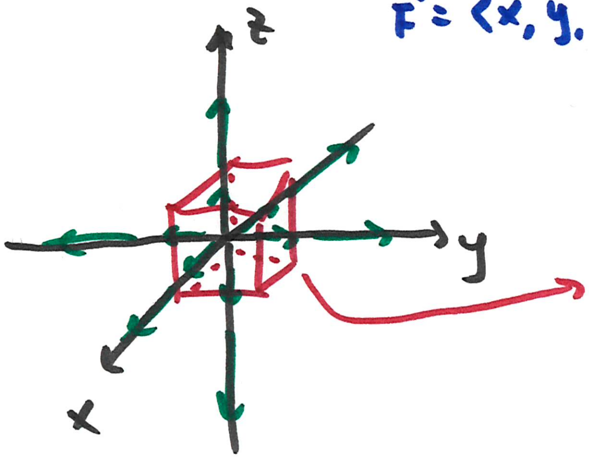
S : the six sides of the cube

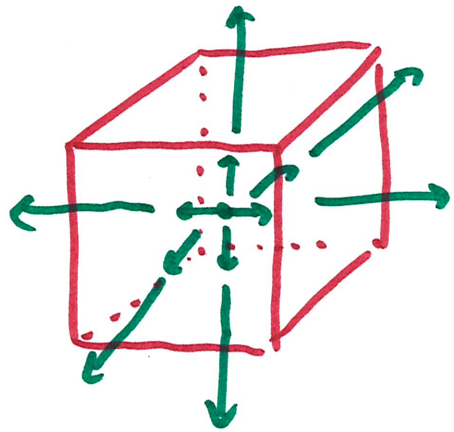
E : volume inside

why is the Theorem true?

divergence can be interpreted as a volume change of a small box in $\vec{F}$

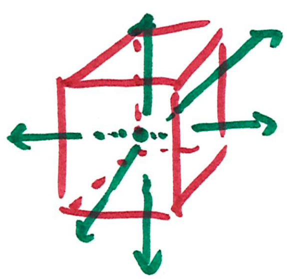
$\vec{F} = \langle x, y, z \rangle$





if box is rigid,
box expands if $\operatorname{div} \vec{F} > 0$
box shrinks if $\operatorname{div} \vec{F} < 0$

if box doesn't change ~~box~~ volume but surface is porous, then
if $\operatorname{div} \vec{F} > 0$ → things flowing out
$\operatorname{div} \vec{F} < 0$ → " " in



flow <u>through</u> sides relate directly to divergence

flux integral $\iint_S \vec{F} \cdot d\vec{S}$

accumulated by $\iiint_E \operatorname{div} \vec{F}\, dV$

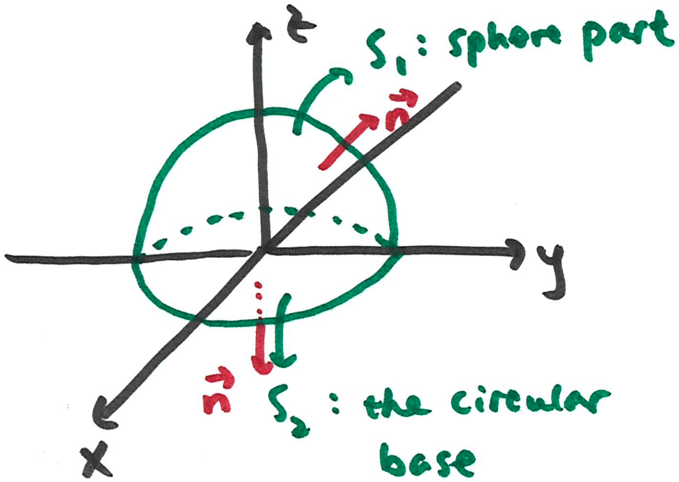
the divergence theorem is usually only used if the surface is closed (no openings)

example $\vec{F} = \langle x, y, z \rangle$

S : upper half of $x^2+y^2+z^2=9$ and the circle enclosing the volume at $z=0$

as usual, $\vec{n}$ is positive pointing outward

Let's verify the theorem: $\iiint_E \operatorname{div}\vec{F}\,dV = \iint_S \vec{F}\cdot d\vec{S}$



outward normal means $\vec{n}$ upward on the sphere part and downward on the circle

surface integral first $\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S}_1 + \iint_{S_2} \vec{F}\cdot d\vec{S}_2$

parametrize S_1: Spherical, $\rho = 3$, so let $u = \phi$, $v = \theta$

$$\vec{r}(u,v) = \langle \underbrace{3\sin u\cos v}_{x=\rho\sin\phi\cos\theta},\ 3\sin u\sin v,\ 3\cos u \rangle \qquad \begin{array}{l} 0 \le u \le \pi/2 \\ 0 \le v \le 2\pi \end{array}$$

$$\vec{r}_u = \langle 3\cos u\cos v,\ 3\cos u\sin v,\ -3\sin u \rangle$$

$$\vec{r}_v = \langle -3\sin u\sin v,\ 3\sin u\cos v,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 9\sin^2 u\cos v,\ 9\sin^2 u\sin v,\ 9\cos u\sin v \rangle$$ direction correct? yes

parametrize S_2: cylindrical / polar $u = r$ $v = \theta$

$$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ 0 \rangle \qquad \begin{array}{l} 0 \le u \le 3 \\ 0 \le v \le 2\pi \end{array}$$

$$\vec{r}_u = \langle \cos v, \sin v, 0 \rangle$$

$$\vec{r}_v = \langle -u\sin v,\ u\cos v,\ 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 0, 0, u \rangle$$ direction correct? No, "out" on S_2 is "down"

use $\vec{r}_v \times \vec{r}_u = \langle 0, 0, -u \rangle$

now $\iint_S \vec{F}\cdot(\underbrace{\vec{r_u}\times\vec{r_v}}_{\text{or } \vec{r_v}\times\vec{r_u}})\,dA$ for each

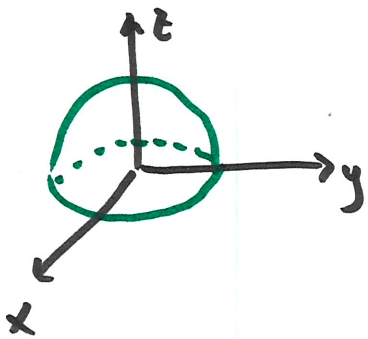
$$\int_0^{2\pi}\int_0^{\pi/2} \underbrace{\langle 3\sin u\cos v,\ 3\sin u\sin v,\ 3\cos u\rangle}_{\vec{F}=\langle x,y,z\rangle \text{ using } \vec{r}(u,v)}\cdot\langle 9\sin^2 u\cos v,\ 9\sin^2 u\sin v,\ 9\cos u\sin u\rangle\, du\,dv$$

$$+\int_0^{2\pi}\int_0^{3}\langle u\cos v,\ u\sin v,\ 0\rangle\cdot\langle 0,\ 0,\ -u\rangle\, du\,dv$$

$$= \dots = \boxed{54\pi}$$

Divergence : $\iiint_E \operatorname{div} \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S}$

now the equivalent triple integral : $\operatorname{div} \vec{F} = \overbrace{\vec{\nabla}}^{\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle} \cdot \langle x, y, z \rangle = 3$



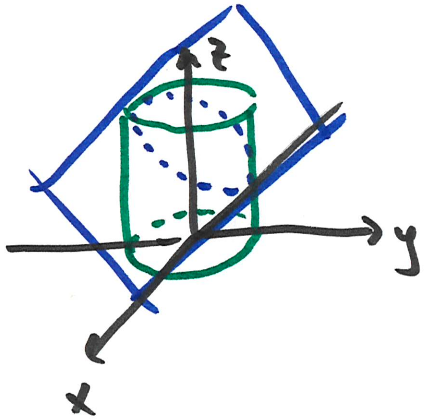
$$\iiint_E 3\, dv = 3 \underbrace{\iiint_E dv}_{\text{volume of enclosed space : half of a sphere radius 3}}$$

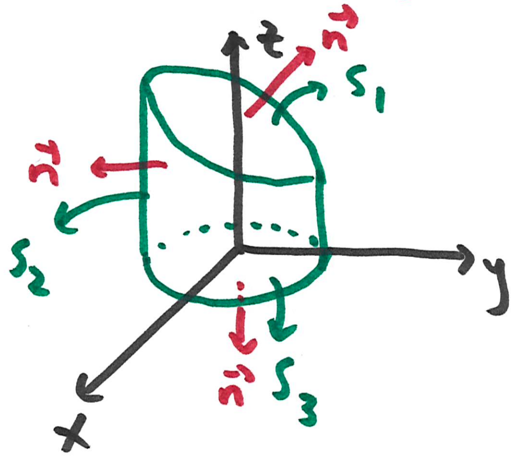
$$= 3 \cdot \frac{1}{2} \cdot \frac{4}{3} \pi (3)^3 = 2\pi \cdot 27 = \boxed{54\pi}$$

<u>example</u> $\vec{F} = \langle -y^3z^2,\ x^4,\ 4xy^2 \rangle$

S: surface of space enclosed by $\underbrace{x^2+y^2=1}_{\text{cylinder}}$, $\underbrace{z=10-y}_{\text{plane}}$, $\underbrace{z=0}_{\text{xy-plane}}$

normal is positive outward





3 surfaces

3 parametrizations

3 normals (3 cross products)

enclosed space, so divergence theorem → one triple integral

$$\iiint_E \operatorname{div}\vec{F}\, d\vec{V}$$

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\langle -y^3z^2,\ x^4,\ 4xy^2\rangle =$$

$$= \frac{\partial}{\partial x}(-y^3z^2) + \frac{\partial}{\partial y}(x^4) + \frac{\partial}{\partial z}(4xy^2)$$

$$= 0$$

$$= \iiint_E \operatorname{div}\vec{F}\, d\vec{V} = \boxed{0}$$