14.3 Motions in space (continued)

$\vec{r}_1(t) = \langle \cos t, \sin t \rangle \qquad 0 \le t \le 2\pi$

$\vec{r}_2(t) = \langle \sin t, \cos t \rangle \qquad 0 \le t \le 2\pi$

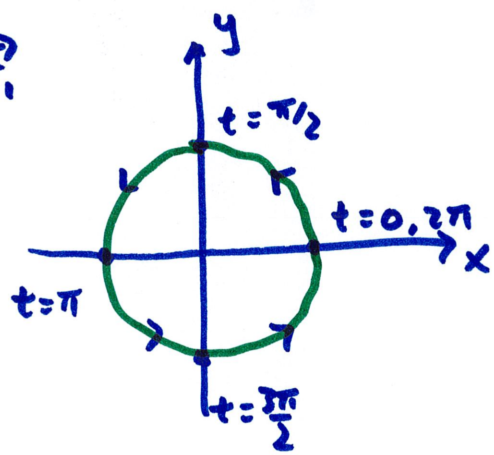
shape ? circle

note for $\vec{r}_1$, $x(t) = \cos t$ $\quad y(t) = \sin t$ $\quad \cos^2 t + \sin^2 t = 1 \rightarrow x^2 + y^2 = 1$

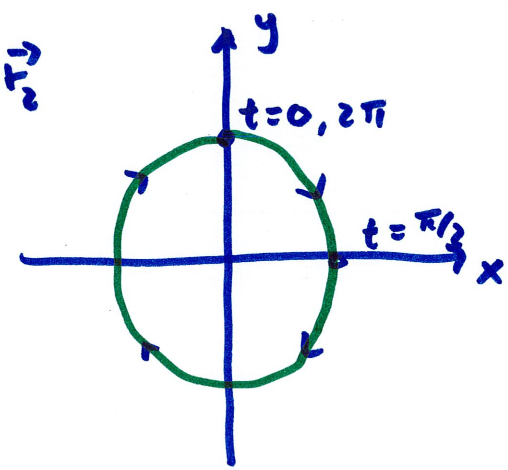
circle radius 1

$\vec{r}_2$ $\quad x(t) = \sin t$ $\quad y(t) = \cos t$ $\quad \cos^2 t + \sin^2 t = 1 \rightarrow y^2 + x^2 = 1$

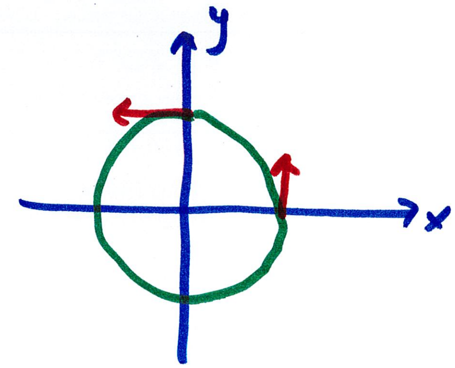
$\vec{r}_1$



$\vec{r}_2$



we can also find direction from velocity



$\vec{r}_1 = \langle \cos t, \sin t \rangle \qquad 0 \le t \le 2\pi$

$\vec{r}_1{}' = \vec{v}_1 = \langle -\sin t, \cos t \rangle \qquad 0 \le t \le 2\pi$

$\vec{v}_1(0) = \langle 0, 1 \rangle$ vector pointing "up"

$\vec{v}_1(\frac{\pi}{2}) = \langle -1, 0 \rangle$ " " left

note $\vec{r}_1 \cdot \vec{v}_1 = \langle \cos t, \sin t \rangle \cdot \langle -\sin t, \cos t \rangle = 0$

so position and velocity vectors are orthogonal for all t

this is true for all circles

in general, if $\vec{r}(t)$ is a circle, $\vec{r}(t) = \langle x(t), y(t) \rangle$

then $x^2 + y^2 = R^2 \qquad R$: some constant (radius)

and $\vec{r} \cdot \vec{r}\,' = 0$

$\vec{r}_3(t) = \langle \underbrace{\sin t + 2\sqrt{6}\cos t}_{x},\ \underbrace{2\sqrt{6}\sin t - \cos t}_{y} \rangle \qquad 0 \le t \le 2\pi$

$x^2 = \sin^2 t + 4\sqrt{6}\sin t\cos t + 24\cos^2 t$

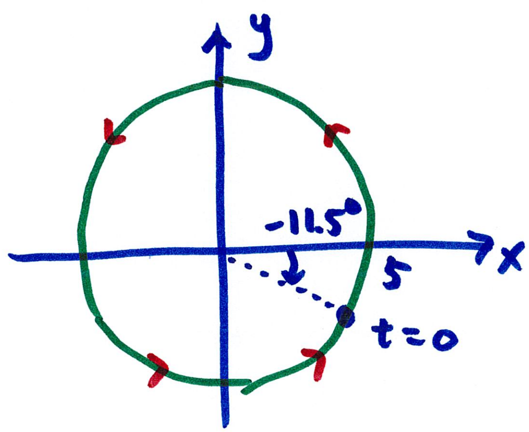
$y^2 = 24\sin^2 t - 4\sqrt{6}\sin t\cos t + \cos^2 t$

$x^2 + y^2 = 25\cos^2 t + 25\sin^2 t = 25$ circle radius 5

$\vec{r}_3\,' = \langle \cos t - 2\sqrt{6}\sin t,\ 2\sqrt{6}\cos t + \sin t \rangle$

note $\vec{r}_3 \cdot \vec{r}_3\,' = 0$

this is a circle with a shifted starting location



because $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

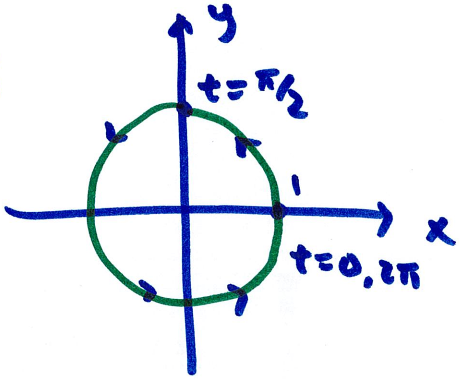
$\vec{r}_1 = \langle \cos t, \ \sin t \rangle \qquad 0 \le t \le 2\pi$

$\vec{r}_4 = \langle \cos(2t), \ \sin(2t) \rangle \qquad 0 \le t \le B$ the "2" is the frequency

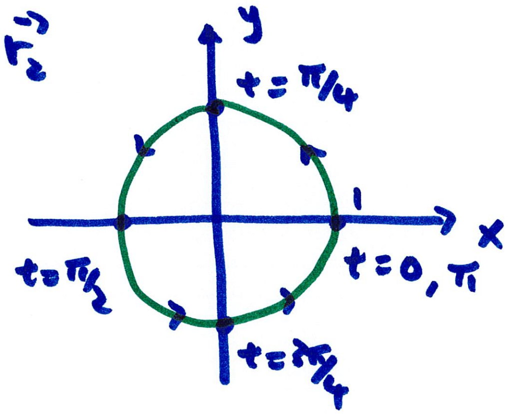
period $= \dfrac{2\pi}{freq}$

$\cos^2(2t) + \sin^2(2t) = 1$ circle radius 1

$\vec{r}_1$



$\vec{r}_2$



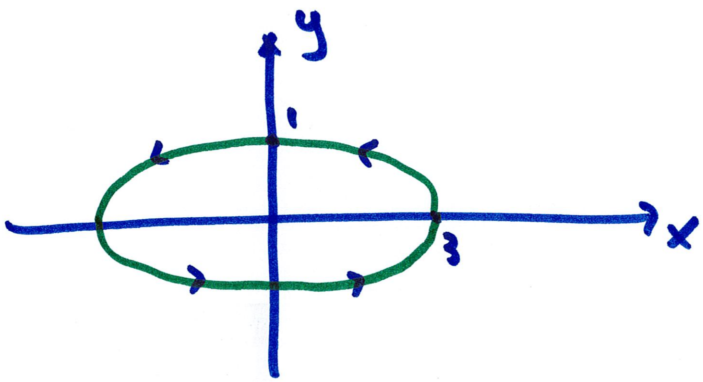
so, $B = \pi$

$\vec{r}_1\,' = \langle -\sin t, \cos t \rangle \qquad |\vec{r}_1\,'| = 1$ speed of $\vec{r}_1$

$\vec{r}_4\,' = \langle -2\sin 2t, \ 2\cos 2t \rangle \qquad |\vec{r}_4\,'| = 2$ twice the speed [illegible] $\vec{r} = \langle 3\cos t,\ \sin t \rangle \qquad 0 \le t \le 2\pi$

ellipse $\quad x = 3\cos t \quad \frac{x}{3} = \cos t \quad \left(\frac{x}{3}\right)^2 + y^2 = 1$

$y = \sin t \quad y = \sin t \quad \frac{x^2}{9} + y^2 = 1$

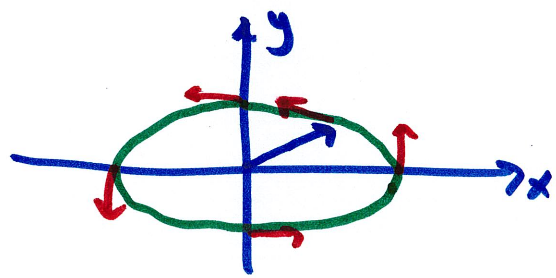


$\vec{v} = \vec{r}\,' = \langle -3\sin t,\ \cos t \rangle \qquad |\vec{v}| = 9\sin^2 t + \cos^2 t = 1 + 8\sin^2 t$

$\vec{r} \cdot \vec{v} = -9\cos t\sin t + \cos t \sin t = -8\cos t \sin t \ne 0$ for all t

also, $|\vec{v}| \ne$ constant

velocity is no longer ALWAYS orthogonal to position



Projectile Motion

For sake of calculation, let $g = 10\ m/s^2$

An object is launched from ground with initial velocity

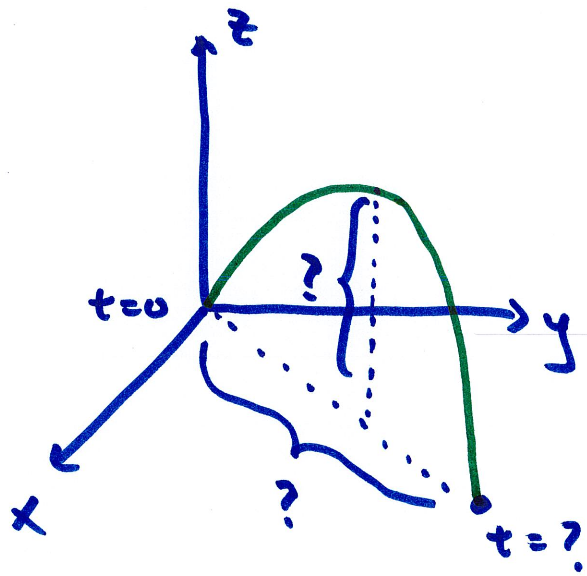
$\vec{v}(0) = \langle 1.25, 2.5, 5 \rangle$ m/s under only gravitational acceleration.

So, $\vec{a}(t) = \langle 0, 0, -10 \rangle$. Assume starting at origin $\vec{r}(0) = \langle 0, 0, 0 \rangle$

Find: time of flight

range (distance covered ground)

max height



$\vec{a}(t) = \langle 0, 0, -10 \rangle$

$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle 0, 0, -10t \rangle + \vec{c} = \langle c_1, c_2, -10t + c_3 \rangle$

use $\vec{v}(0) = \langle 1.25, 2.5, 5 \rangle$ to find c_1, c_2, c_3

$\vec{v}(0) = \underbrace{\langle c_1, c_2, 0 + c_3 \rangle}_{\text{from my } \vec{v}(t) \text{ from } \int \vec{a}(t)\,dt} = \underbrace{\langle 1.25, 2.5, 5 \rangle}_{\text{given}}$ so $c_1 = 1.25$, $c_2 = 2.5$, $c_3 = 5$

so, $\boxed{\vec{v}(t) = \langle 1.25, 2.5, -10t + 5 \rangle}$

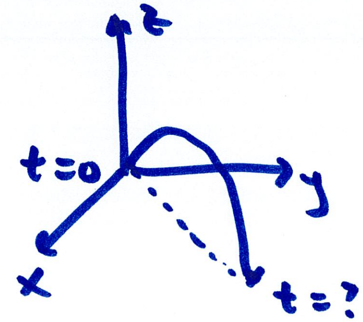
$\vec{r}(t) = \int \vec{v}(t)\,dt = \langle 1.25t + d_1, 2.5t + d_2, -5t^2 + 5t + d_3 \rangle$

use $\vec{r}(0) = \langle 0, 0, 0 \rangle$ to find d_1, d_2, d_3

we see $d_1 = 0$, $d_2 = 0$, $d_3 = 0$

so, $\boxed{\vec{r}(t) = \langle 1.25t, 2.5t, -5t^2 + 5t \rangle}$

Time of flight ?

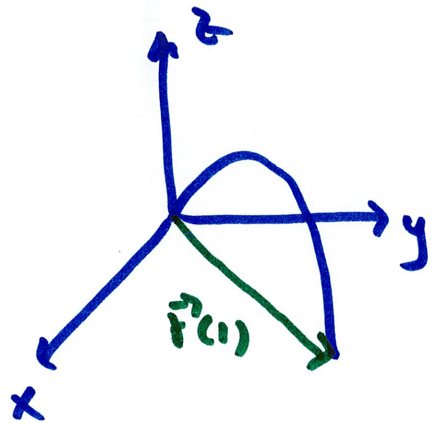


z-component of position : $-5t^2+5t=0$ (ground)

$-5t(t-1)=0$ $\quad t=0,\ t=1$

so time of flight is $\boxed{1 \text{ second}}$

Range ?



$\vec{r}(1) = \langle 1.25,\ 2.5,\ 0 \rangle$

$|\vec{r}(1)| \approx \boxed{2.8 \text{ meters}}$

Max height ?

→ when vertical component of $\vec{v}$ is 0

$$\vec{v}_z = -10t + 5 = 0 \rightarrow t = \frac{1}{2}$$

height : z of position at $t = \frac{1}{2}$ → $\boxed{1.25 \text{ meters}}$

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what if $\vec{v}(0)$ is doubled ?

new $\vec{v}(0) = \langle 2.5, 5, 10 \rangle$ same $\vec{r}(0) = \langle 0, 0, 0 \rangle$

$$\vec{a}(t) = \langle 0, 0, -10 \rangle$$

$$\vec{v}(t) = \int \vec{a}(t)\,dt = \langle c_1, c_2, -10t + c_3 \rangle$$

$$= \langle 2.5, 5, -10t + 10 \rangle$$

$$\vec{r}(t) = \int \vec{v}(t)\,dt = \dots = \langle 2.5t, 5t, -5t^2 + 10t \rangle$$

time of flight : $-5t^2+10t=0$

$-5t(t-2)=0$ $\quad t=0,\ t=2$ **doubled**

range : $|\vec{r}(2)| = \dots = 11.18$ **quadrupled**