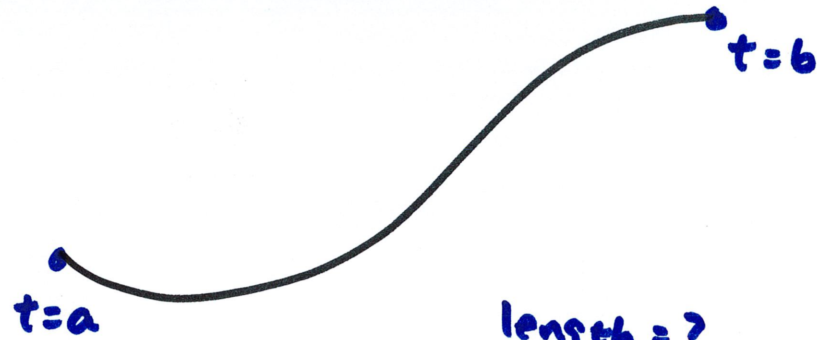
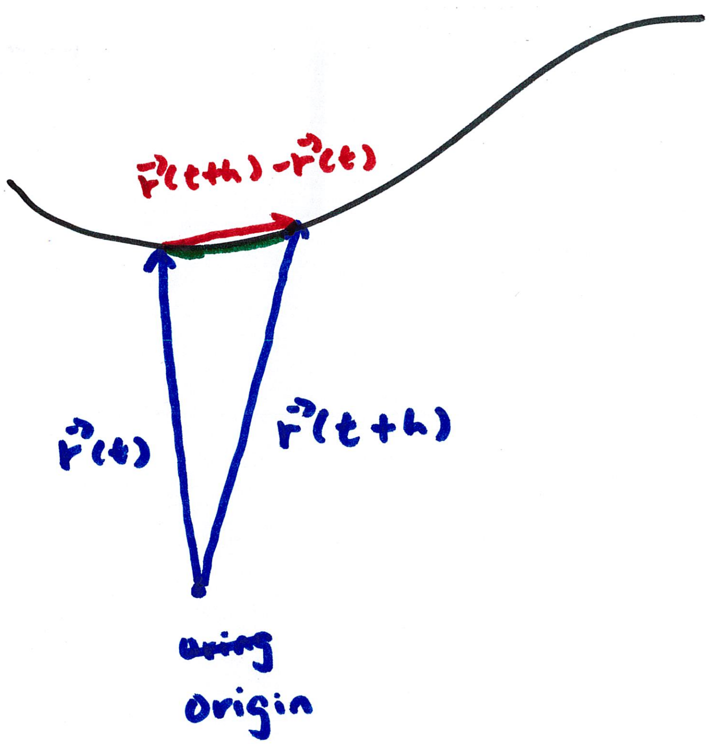
14.4 Length of Curves

$\vec{r}(t)$ $\quad a \le t \le b$



length = ?



note if h is small,

then $|\vec{r}(t+h)-\vec{r}(t)| \approx$ actual length from t to $t+h$

how to calculate $|\vec{r}(t+h)-\vec{r}(t)|$ to estimate small segment of length?

once done, accumulate by integration

recall $\vec{r}\,'(t) = \lim_{h \to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$

this means when h is small

$$\vec{r}\,'(t) \approx \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

so $\vec{r}(t+h) - \vec{r}(t) \approx h\,\vec{r}\,'(t)$

$$|\vec{r}(t+h) - \vec{r}(t)| \approx h\,|\vec{r}\,'(t)|$$

So, one small segment has length given by $|\vec{r}\,'(t)|\,h$

total length → L = sum of all segments $\approx \sum_{i=1}^{n} |\vec{r}\,'(t)|\,h$ 　 h is change in t call it Δt

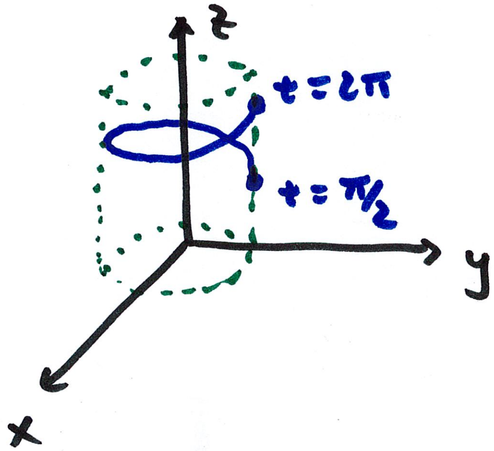
$$\approx \sum_{i=1}^{n} |\vec{r}\,'(t)|\,\Delta t$$

now let $n \to \infty$ so $\sum_{i=1}^{\infty} \to \int$, $\Delta t \to dt$

therefore, $\boxed{L = \int_a^b |\vec{r}\,'(t)|\,dt}$ this is the length (exact) on $a \le t \le b$

example $\vec{r}(t) = \langle \cos t, \sin t, t \rangle \qquad \frac{\pi}{2} \le t \le 2\pi$

helix traveling on surface of cylinder



$$L = \int_a^b |\vec{r}\,'(t)|\,dt$$

$$\vec{r}\,' = \langle -\sin t, \cos t, 1 \rangle$$

$$|\vec{r}\,'| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\int_{\pi/2}^{2\pi} \sqrt{2}\,dt = \sqrt{2}\,t \Big|_{\pi/2}^{2\pi} = \sqrt{2}\left(2\pi - \pi/2\right) = \boxed{\frac{3\pi}{2}\sqrt{2}}$$

what if we want the length as a function of t? $\quad s(t) = ?$ $\quad a \le t \le$ something else

modify $\quad L = \int_a^b |\vec{r}\,'(t)|\,dt$

remove b, replace it with t

$$\boxed{s(t) = \int_a^t |\vec{r}\,'(u)|\,du}$$

u is called a "dummy variable"
we don't want t to be the variable when we integrate it to t

try it on the helix $\quad \vec{r}(t) = \langle \cos t, \sin t, t \rangle$

$$|\vec{r}\,'| = \sqrt{2}$$

$$s(t) = \int_{\pi/2}^{t} \sqrt{2}\,du = \sqrt{2}\,u \Big|_{\pi/2}^{t} = \boxed{\sqrt{2}\,(t - \pi/2)}$$

at $t = 2\pi \qquad s(2\pi) = \sqrt{2}\,(2\pi - \pi/2) = \frac{3\pi}{2}\sqrt{2}$ $\quad$ (same as before)

$s(t) = \int_a^t |\vec{r}\,'(u)|\,du$ gives relationship between s and t

"distance" "time"

in $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ gives us the position at a given time (t)

but we know $s(t)$ so we can change the parameter from time (t) to distance (s) to get $\vec{r}(s) \to$ location having traveled distance of s

back to helix : $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ $\quad \pi/2 \le t \le 2\pi$

we found $s(t) = \sqrt{2}(t - \pi/2)$ from $t = \pi/2$ to some other t

$$s = \sqrt{2}(t - \pi/2) \to t = \frac{s}{\sqrt{2}} + \frac{\pi}{2}$$

sub in $\vec{r}(t)$:

$$\boxed{\vec{r}(s) = \left\langle \cos\left(\frac{s}{\sqrt{2}} + \frac{\pi}{2}\right), \sin\left(\frac{s}{\sqrt{2}} + \frac{\pi}{2}\right), \frac{s}{\sqrt{2}} + \frac{\pi}{2} \right\rangle}$$

"using arc length as parameter"

$\pi/2 \le t \le 2\pi \to$ length at $t = \pi/2$ $\le s \le$ length at $t = 2\pi$

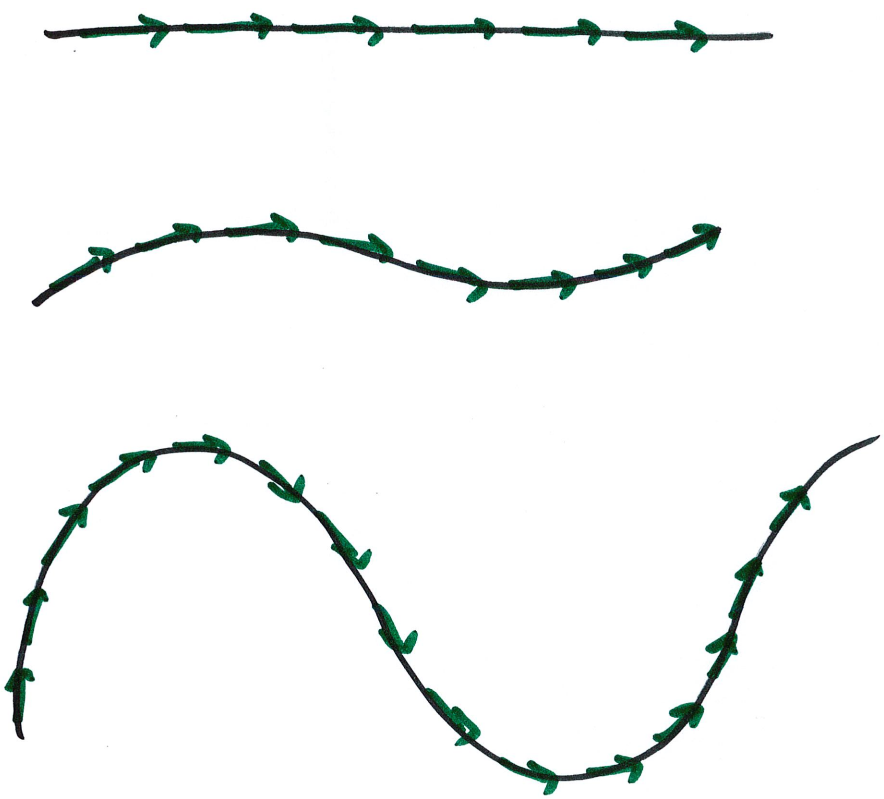
$$0 \le s \le \frac{3\pi}{2}\sqrt{2}$$

14.5 Curvature

curvature is a measure of how "curvy" a curve is

Straight line has curvature of 0

given $\vec{r}(t)$, how to find curvature $\kappa(t)$ "kappa"



recall $\vec{T}$ is the unit tangent vector

$$\vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

the unit tangent vector $\vec{T}$ changes more rapidly on more curvy portion

so we need to track the change in $\vec{T}$ as we move along

$\rightarrow$ $\boxed{\left|\frac{d\vec{T}}{ds}\right| = \kappa}$

length

but $\frac{d\vec{T}}{ds}$ is not not always most practical, since $\vec{T}$ is usually

function of t (because $\vec{r}(t)$ is more often given as function of t)

→ find alternative formula that deals with t

$$s(t) = \int_a^t |\vec{r}\,'(u)|\,du$$

we get $\frac{d}{dt} s(t) = \frac{d}{dt} \int_a^t |\vec{r}\,'(u)|\,du$

$$\frac{ds}{dt} = |\vec{r}\,'(t)|$$ from the Fundamental Theorem of Calculus (part 1)

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{|d\vec{T}|}{|ds|} = \frac{\left|\frac{d\vec{T}}{dt}\right|}{\left|\frac{ds}{dt}\right|} = \boxed{\frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \kappa}$$

other forms (vector) : $\kappa = \frac{|\vec{r}\,'' \times \vec{r}\,'|}{|\vec{r}\,'|^3}$