Variation of parameters

$$y'' - 4y = \cosh(2t)$$

complementary: $y'' - 4y = 0 \rightarrow y_c = c_1 e^{2t} + c_2 e^{-2t}$

$y_1 = e^{2t} \quad y_2 = e^{-2t}$

$$y = u_1 y_1 + u_2 y_2$$

solve $\quad u_1' y_1 + u_2' y_2 = 0$

$\quad u_1' y_1' + u_2' y_2' = f(t) \leftarrow$ right side of equation

$$\begin{bmatrix} e^{2t} & e^{-2t} & 0 \\ 2e^{2t} & -2e^{-2t} & \cosh(2t) \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} e^{2t} & e^{-2t} & 0 \\ 0 & -4e^{-2t} & \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t} \end{bmatrix}$$

$$-4e^{-2t}u_2' = \frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}$$

$$u_2' = -\frac{1}{8}e^{4t} - \frac{1}{8} \qquad u_2 = -\frac{1}{32}e^{4t} - \frac{1}{8}t + c_2$$

$$e^{2t}u_1' + e^{-2t}u_2' = 0$$

$$e^{2t}u_1' = -e^{-2t}u_2' = -e^{-2t}\left(-\frac{1}{8}e^{4t} - \frac{1}{8}\right) = \frac{1}{8}e^{2t} + \frac{1}{8}e^{-2t}$$

$$u_1' = \frac{1}{8} + \frac{1}{8}e^{-4t} \qquad u_1 = \frac{1}{8}t - \frac{1}{32}e^{-4t} + c_1$$

general solution

$$y = \left(\frac{1}{8}t - \frac{1}{32}e^{-4t} + c_1\right)e^{2t} + \left(-\frac{1}{32}e^{4t} - \frac{1}{8}t + c_2\right)e^{-2t}$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ eigenvalues $$\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 = 0 \qquad \lambda = 1, 1$$

eigenvectors: $\underline{\lambda = 1}$ $\quad (A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

two free variables

$x_2 = r \qquad x_1 = t$

$$\vec{v} = \begin{bmatrix} t \\ r \end{bmatrix} = t \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} + r \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

two indp vectors
that span the eigenspace
for $\lambda = 1$
two basis vectors
$\rightarrow$ dimension two eigenspace

null space → space spanned by solution to $A\vec{x} = \vec{0}$

solve, then find basis vectors

the number of basis vectors is called the nullity

example $A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}$

solve $A\vec{x} = \vec{0}$ $\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 8 & 0 \end{bmatrix}$$

$$\cdots \rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad x_2 = 0,\ x_1 = 0$$

solution is one point: $x_1 = 0,\ x_2 = 0$

does NOT span more than one point

→ no basis vector → dimension (nullity) is zero example

$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$

solve $A\vec{x} = \vec{0}$

$\begin{bmatrix} 1 & 2 & 3 & 0 \end{bmatrix}$

one pivot, three variables → two free

$x_3 = r, \quad x_2 = t$

$x_1 + 2x_2 + 3x_3 = 0$

$x_1 = -2x_2 - 3x_3 = -2t - 3r$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t - 3r \\ t \\ r \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + r\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

two basis vectors: $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

dimension is 2

note: $(\#\text{ variables}) - \underbrace{(\#\text{ pivots})}_{\text{"rank"}} = \text{dimension of nullspace}$