Let $y(x)$ be the solution of the following initial value problem

$$y'(x) = y\cos x, \quad y(0) = 1.$$

Then $y(\frac{\pi}{2})$ is equal to

A. $y(\frac{\pi}{2}) = 1$

B. $y(\frac{\pi}{2}) = e^2$

C. $y(\frac{\pi}{2}) = e$

D. $y(\frac{\pi}{2}) = 2e$

E. $y(\frac{\pi}{2}) = 3e$

Separable?

$\frac{dy}{dx} = y\cos x$ yes.

$\frac{1}{y}\,dy = \cos x\,dx$

$\int \frac{1}{y}\,dy = \int \cos x\,dx$

$\ln|y| = \sin x + c$

$y = e^{\sin x + c} = e^{\sin x} \cdot e^{c}$

$y = Ce^{\sin x} \longrightarrow \boxed{y = e^{\sin x}}$

$y(0) = 1$

$1 = Ce^{0} = C$

$y(\frac{\pi}{2}) = e^{1} = e$

Let $y(t)$ be the solution to the initial value problem

$$\frac{dy}{dt} - 2y = 3e^{-t}, \quad y(0) = 7$$

Find T such that $y(T) = 0$.

$y' - 2y = 3e^{-t}$

$y' + p(t)y = g(t)$ 1st-order linear

find integrating factor

$I = e^{\int p(t)dt} = e^{\int -2dt} = e^{-2t}$

$\underbrace{e^{-2t}y' - 2ye^{-2t}} = e^{-2t} \cdot 3e^{-t}$

$\frac{d}{dt}(e^{-2t}y) = 3e^{-3t}$

$e^{-2t}y = \int 3e^{-3t}dt$

A. $T = \frac{1}{2}\ln 3$

B. $T = -\frac{1}{2}\ln 3$

C. $T = \frac{1}{3}\ln 8$

D. $T = -\frac{1}{3}\ln 8$

E. $T = \frac{1}{2}\ln 2$

$$e^{-2t} y = -e^{-3t} + C$$

$$y = (-e^{-3t} + C)e^{2t} = -e^{-t} + Ce^{2t}$$

$$y(0) = 7 \longrightarrow 7 = -1 + C \qquad C = 8$$

$$\boxed{y(t) = -e^{-t} + 8e^{2t}}$$

$$y(T) = -e^{-T} + 8e^{2T} = 0$$

$$8e^{2T} = e^{-T} \qquad \text{multiply by } e^{T}$$

$$8e^{3T} = 1$$

$$e^{3T} = \frac{1}{8}$$

$$3T = \ln\left(\frac{1}{8}\right) \qquad T = \frac{1}{3}\ln\left(\overset{8^{-1}}{\frac{1}{8}}\right) = -\frac{1}{3}\ln(8)$$

Let $y(x)$ satisfy the following exact equation with initial condition

$$(2x^2e^{2y} - 3x\cos y)dy + (-3\sin y + 2xe^{2y} + 2)dx = 0, \quad y(1) = 0.$$

Which of the following give an implicit formula for $y(x)$?

A. $2y - x^2e^{2y} + 3x\sin y = -1$

B. $2y + x^2e^{2y} + 3x\sin y = 1$

C. $x^2 + x^2e^{2y} + 3x\sin y = 2$

D. $2x + x^2e^{2y} - 3x\sin y = 3$

E. $2x - x^2e^{2y} - 3x\sin y = 1$

exact

$M\,dx + N\,dy = 0$

such that $M_y = N_x$

solution: $f(x,y) = c$

where $f_x = M$ and $f_y = N$

$f_x = -3\sin y + 2xe^{2y} + 2$

$f_y = 2x^2e^{2y} - 3x\cos y$

pick one to integrate, here $f_x = M$

$$f = \int \left(-3\sin y + 2xe^{2y} + 2\right) d\underline{x} \quad \text{y is constant}$$

$$= -3\sin y \cdot x + x^2e^{2y} + 2x + g(y)$$

take partial with y, compare to N to find g(y)

$$f_y = -3x\cos y + 2x^2e^{2y} + \frac{dg}{dy} = 2x^2e^{2y} - 3x\cos y$$

$$\text{so } \frac{dg}{dy} = 0, \text{ so } g = C$$

so, solution is $f(x,y) = C$

$$-3x\sin y + x^2e^{2y} + 2x = C \qquad y(\underset{x}{1}) = 0$$

$$\cancel{0 = C}$$

$$\cancel{-3x\sin y + x^2e^{2y} + 2x =}$$

$$-3\sin(0) + (1)e^0 + 2(1) = C$$

$$C = 3$$

$$\boxed{-3x\sin y + x^2e^{2y} + 2x = 3}$$

Let A be a 3×3 matrix and let E be the reduced row-echelon form of A. If the second row of E is $[0 \quad 1 \quad -1]$, consider the following statements:

I. A is not row-equivalent to the 3×3 identity matrix.
II. It is possible that the rank of A is equal to one.
III. The rank of A is equal to two.

$$A = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}$$

A. I is the only true statement

B. II is the only true statement

C. I and II are the only true statements

D. II and III are the only true statements

E. I and III are the only true statements

reduce row-echelon

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

elements above and below pivots are zero and pivot of row is below and to the right of the row above.
row of zeros at bottom.

reduce row echelon of A is

$$\begin{bmatrix} 1 & 0 & ? \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

pivot in row 2 eliminates this

→ pivot in col 2 so pivot above in col 1 otherwise row 1 is all zeros can't be row 1

can't be a pivot otherwise the element above has to be zero

I. cannot change A into I by row ops. True

II. False

III. "rank" → # of pivots True

Find all constants a such that the origin is a saddle point for the system

$$X'(t) = \begin{bmatrix} a & 2 \\ 2 & a \end{bmatrix} X(t), \quad a \text{ in } \mathbb{R},$$

A. $a < 2$

B. $a > -2$

C. $a < -3$ or $a > 3$

D. $a < -2$ or $a > 2$

E. $-2 < a < 2$

saddle point : eigenvalues of $\begin{bmatrix} a & 2 \\ 2 & a \end{bmatrix}$ are real and have opposite signs

$$\begin{vmatrix} a-\lambda & 2 \\ 2 & a-\lambda \end{vmatrix} = 0$$

$(a-\lambda)^2 - 4 = 0$

$(a-\lambda)^2 = 4$

$a - \lambda = 2$ or $a - \lambda = -2$

$\lambda = a-2$ or $\lambda = a+2$

$\hookrightarrow$ better way : $a^2 - 2a\lambda + \lambda^2 - 4 = 0$

$$\lambda^2 - 2a\lambda + (a^2-4) = 0$$

$$\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2-4)}}{2}$$

$$= \frac{2a \pm \sqrt{16}}{2} = a \pm 2$$

Find the Wronskian determinant $W(f, g, h)$ for the three functions

$$f(x) = x, \quad g(x) = e^x, \quad h(x) = e^{2x}.$$

$$W = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$$

$$= \begin{vmatrix} x & e^x & e^{2x} \\ 1 & e^x & 2e^{2x} \\ 0 & e^x & 4e^{2x} \end{vmatrix} \quad \text{expand along col 1}$$

A. $-e^{3x}$

B. $x(2x+3)e^{3x}$

C. $(2x+3)e^{3x}$

D. $2x-3$

E. $(2x-3)e^{3x}$

$$= (x)\begin{vmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{vmatrix} - (1)\begin{vmatrix} e^x & e^{2x} \\ e^x & 4e^{2x} \end{vmatrix}$$

$$= (x)(4e^{3x} - 2e^{3x}) - (4e^{3x} - e^{3x})$$

$$= 2xe^{3x} - 3e^{3x} = (2x-3)e^{3x}$$