5. Let $y(x)$ satisfy the following second order differential equation

$$yy'' = 3(y')^2,$$
$$y(0) = 1, \quad y'(0) = 1.$$

Then $y(\frac{3}{8})$ is equal to

A. $y(\dfrac{3}{8}) = 1$

B. $y(\dfrac{3}{8}) = 2$

C. $y(\dfrac{3}{8}) = 3$

D. $y(\dfrac{3}{8}) = 4$

E. $y(\dfrac{3}{8}) = 5$

Solve by a substitution

$yy'' = 3(y')^2$ no x here

let $p = \frac{dy}{dx} = y'$

$p' = \frac{d^2y}{dx^2} = y'' = \frac{dy}{\cancel{dp}}\frac{\cancel{dp}}{dx} = \frac{dp}{dy}\frac{dy}{dx}$

$= \frac{dp}{dy} p$

$yy'' = 3(y')^2$

$yp\frac{dp}{dy} = 3p^2$

separable in p and y

$$\frac{1}{p}\,dp = \frac{3}{y}\,dy$$

$$\ln p = 3\ln y + C$$

$$p = e^{\ln y^3 + C} = c y^3 = y' \rightarrow \underbrace{\frac{dy}{dx} = c y^3} \rightarrow 1 = c$$

at $x=0,\ y'=1$

$x=0,\ y=1$

$$\frac{1}{y^3}\,dy = c\,dx$$

$$-\frac{1}{2y^2} = cx + D \qquad y(0)=1 \qquad y'(0)=1$$

$$y^2 = \frac{-1}{2cx+2D} = \frac{-1}{2x+2D}$$

$$1 = \frac{-1}{2D} \qquad D = -\frac{1}{2}$$

$$y^2 = \frac{-1}{2x-1} \qquad y = \sqrt{\frac{1}{1-2x}} = \frac{1}{\sqrt{1-2x}}$$

$$y\left(\tfrac{3}{8}\right) = \frac{1}{\sqrt{1-\frac{6}{8}}} = \frac{1}{\sqrt{\frac{2}{8}}} = \frac{1}{\sqrt{\frac{1}{4}}} = \frac{1}{\frac{1}{2}} = 2$$

4. The solution of the differential equation

$$\frac{dy}{dx} = \frac{x^3 + 4y^3}{3xy^2}, \quad x > 0$$

is defined implicitly by the following equation:

A. $x^3 + y^3 - Cxy^3 = 0$

B. $(x + y)^3 - Cx = 0$

C. $x^3 + y^3 - Cx^4 = 0$

D. $x + y - Cx^2 = 0$

E. $1 + y^3 - Cx = 0$

not separable

homogeneous?

↳ as function of $\frac{y}{x}$

$$\frac{dy}{dx} = \frac{x^3 + 4y^3}{3xy^2} \quad \text{divide by } x^3$$

$$= \frac{1 + 4\left(\frac{y}{x}\right)^3}{3\left(\frac{y}{x}\right)^2} \quad \text{yes.}$$

let $v = \frac{y}{x}$

then $y = vx \quad y' = v + xv'$

$$v + xv' = \frac{1+4v^3}{3v^2}$$

v from left side

$$xv' = \frac{1+4v^3}{3v^2} - \frac{3v^3}{3v^2} = \frac{1+v^3}{3v^2}$$

Separable in v, x

$$\frac{3v^2}{1+v^3}\,dv = \frac{1}{x}\,dx$$

$$\ln(1+v^3) = \ln x + C$$

$$1+v^3 = Cx$$

$$v^3 = Cx - 1$$

$$\left(\frac{y}{x}\right)^3 = Cx - 1$$

$$y^3 = Cx^4 - x^3$$

$$x^3 + y^3 - Cx^4 = 0$$

Given that the general solution of the homogeneous equation

$$y^{(4)} + 3y^{(3)} + 3y'' + y' = 0 \text{ is } y_h(x) = C_1 + C_2e^{-x} + C_3xe^{-x} + C_4x^2e^{-x}$$

the general solution to the corresponding nonhomogeneous equation

$$y^{(4)} + 3y^{(3)} + 3y'' + y' = 6x\cos x + 6xe^{-x}$$

looks like:

A. $y(x) = y_h(x) + (Ax + B)\cos x + (Cx + D)\sin x + x^3(Ex + F)e^{-x}$

B. $y(x) = y_h(x) + (Ax + B)\cos x + (Cx + D)\sin x + x^2(Ex + F)e^{-x}$

C. $y(x) = y_h(x) + (Ax + B)\cos x + (Cx + D)\sin x + x(Ex + F)e^{-x}$

D. $y(x) = y_h(x) + x(Ax + B)\cos x + x(Dx + E)\sin x + x^4(Fx + G)e^{-x}$

E. $y(x) = y_h(x) + x(Ax + B)\cos x + x(Cx + D)\sin x + x^3(Ex + F)e^{-x}$

right side: $6x\cos x + 6xe^{-x}$

undetermined coefficients

$$y_p = \overbrace{(Ax+B)\cos x + ((Cx+D)\sin x}^{6x\cos x} + \underbrace{(Ex+F)e^{-x}}_{6xe^{-x}}$$

now check for duplication and adjust

$$y_c = c_1 + \underbrace{c_2 e^{-x}}_{\text{duplicated by } Fe^{-x}} + \underbrace{c_3 x e^{-x}}_{\text{dup. by } Exe^{-x}} + \underbrace{c_4 x^2 e^{-x}}$$

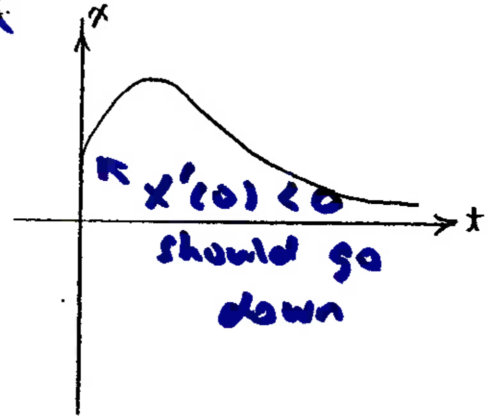
dup. if we multiplied $(Ex+F)e^{-x}$ by x or x^2 to adjust for the other dups.

we need x^3 multiplied to $(Ex+F)e^{-x}$ to avoid duplication

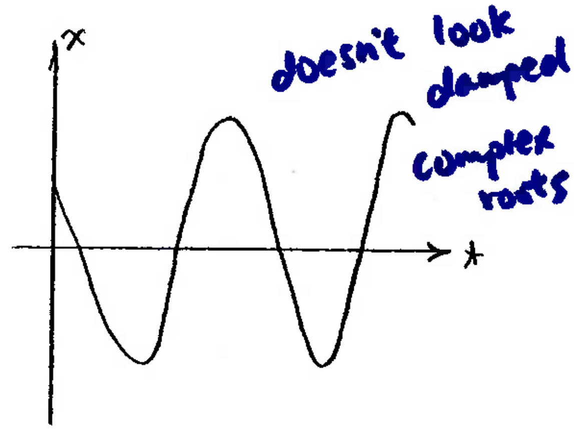
NAME ____________________

21. The oscillation of a spring-mass system is determined by $\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0$, with initial conditions $x(0) = 1$ and $\frac{dx}{dt}(0) = -3$. Then a sketch of the motion $x(t)$ is

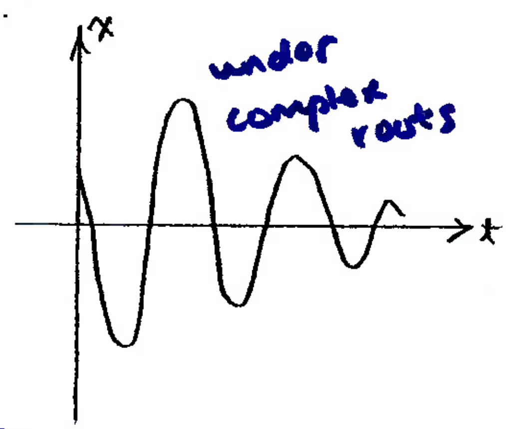
X



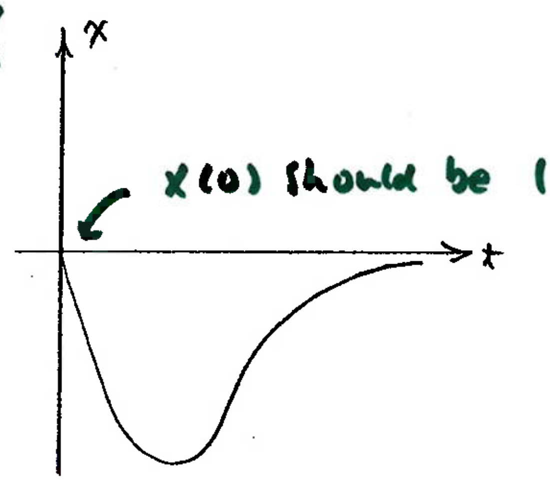
B.



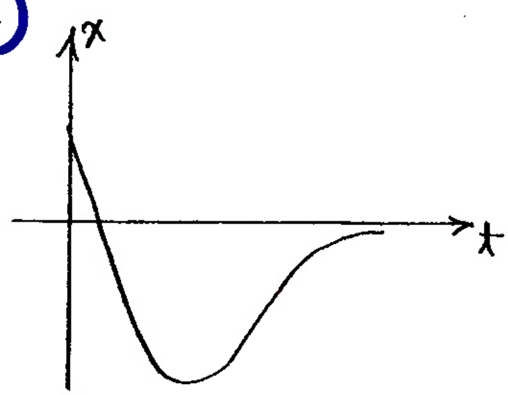
C.



X



E.



$$x'' + 3x' + 2x = 0$$

characteristic eq. : $r^2 + 3r + 2 = 0$

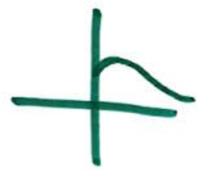
$$(r+2)(r+1) = 0$$

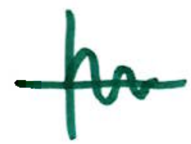
$$r = -2, \quad r = -1$$

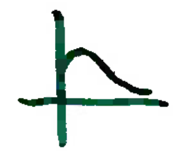
type of damping?

over, under, critically

distinct roots	complex roots	repeated roots
$c^2 > 4km$	$c^2 < 4km$	$c^2 = 4km$







$$mx'' + cx' + kx = 0$$

$$mr^2 + cr + k = 0$$

$$r = \frac{-c \pm \sqrt{\underline{c^2 - 4km}}}{2m}$$

$\lambda = 1$ is an eigenvalue of multiplicity 3 of the matrix $\begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix}$. The defect of the eigenvalue $\lambda = 1$ is equal to

repeated 3 times

# of missing eigenvectors

A. 0

B. 1

C. 2

D. 3

E. 4

$(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{bmatrix} -3 & -9 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 3 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

one pivot

two free variables

↳ two linearly independent vectors to span eigenspace

so, Two true eigenvectors