missing eigenvector(s)

Given that $\lambda = 1$ is a defective eigenvalue of the matrix $\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$, which of the following is the solution of the initial value problem:

$$\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{x} \quad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}?$$

A. $\mathbf{x}(t) = 2e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^t \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}$

B. $\mathbf{x}(t) = 2e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2te^t \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

C. $\mathbf{x}(t) = 2e^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2e^t \left\{ t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$

D. $\mathbf{x}(t) = 2e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^t \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

E. $\mathbf{x}(t) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^t \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

$(A-\lambda I)\vec{v} = \vec{0}$ for true eigenvector

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

true eigenvector $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

missing one $\to$ defect one

need $\vec{v_2}$: let $\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

find $\vec{v_2}$ such that $(A - \lambda I)\vec{v_2} = \vec{v_1}$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 1 & -1 & 1 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$x_2 = \text{free} = r$

$x_1 - x_2 = 1 \qquad x_1 = 1 + x_2 = 1 + r$

so, $\vec{v_2} = \begin{bmatrix} 1+r \\ r \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

choose $r = 0$ so, $\vec{v_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution: $\vec{x}(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^t \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$\vec{x}(0) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 4 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_1 \end{bmatrix}$

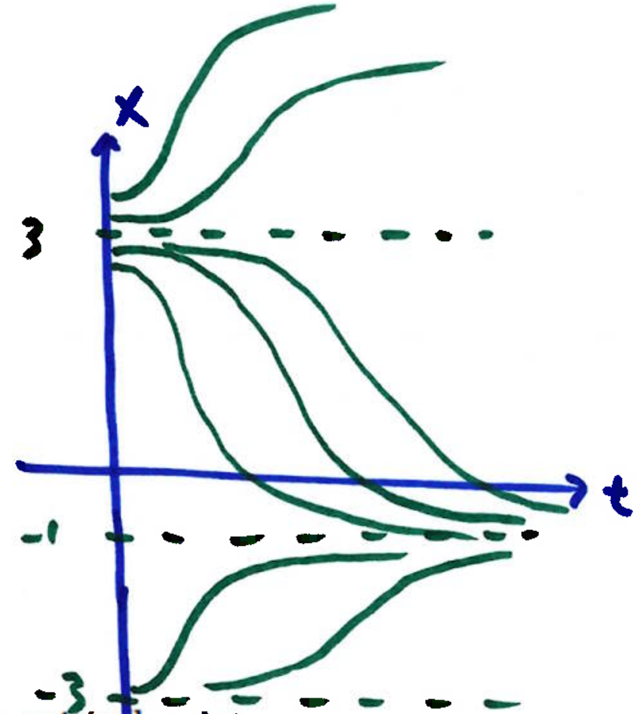
$c_1 = 2$ and $c_2 = 2$

$$\vec{x}(t) = 2e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2e^t \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Consider the autonomous differential equation

$$\frac{dx}{dt} = \left(x^2 - 9\right)(x + 1).$$

Which one of the following statements is **TRUE**?



A. $x = 3$ and $x = -3$ are **stable** critical points.

B. $x = 3$ and $x = -3$ are **semistable** critical points.

C. One of $x = -1$, $x = 3$ is a **semistable** critical point.

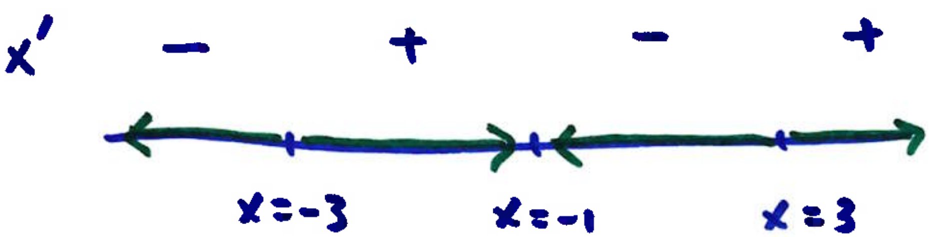
D. $x = -1$ is an **unstable** critical point.

E. $x = -3$ is an **unstable** critical point and $x = -1$ is a **stable** critical point.

critical pts: $\frac{dx}{dt} = 0$

$(x^2 - 9)(x+1) = 0$

so, $x = -3, -1, 3$



$x = -3$: unstable

$x = -1$: stable

$x = 3$: unstable

19. Which of the following statement(s) is/are TRUE?

(I) The vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for $A = \begin{bmatrix} 1 & 3 \\ -3 & -5 \end{bmatrix}$. F

(II) The origin is a *saddle point* for the linear system $\mathbf{x}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}\mathbf{x}$. T

(III) Given that the general solution to the linear system $\mathbf{x}' = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}\mathbf{x}$ is T

$$\mathbf{x}(t) = C_1 e^{3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + C_2 e^{6t}\begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

the origin is a *nodal source.*

I. find eigenvectors of $\begin{bmatrix} 1 & 3 \\ -3 & -5 \end{bmatrix}$

or see if $A\vec{v} = \lambda\vec{v}$ (constant times $\vec{v}$)

$$\begin{bmatrix} 1 & 3 \\ -3 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ 1 \end{bmatrix}?$$

no, so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is NOT an eigenvector

A. Only (II)

B. Only (I) and (III)

C. Only (III)

D. Only (II) and (III)

E. All are TRUE

II. $\vec{x}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \vec{x}$

origin saddle pt if eigenvalues of $\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$ are real and of opposite signs

$$\begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} = 0 \qquad (2-\lambda)^2 - 16 = 0$$

$(2-\lambda)^2 = 16 \qquad 2-\lambda = 4$ or $2-\lambda = -4$

$\lambda = -2$ or $\lambda = 6$

λ

IV. $\vec{x}' = \begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix} \vec{x} \qquad \vec{x}(t) = c_1 e^{\boxed{3}t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{\boxed{6}t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

origin is nodal <u>source</u> if eigenvalues of $\begin{bmatrix} 5 & 2 \\ 1 & 4 \end{bmatrix}$ are positive

$\lambda = 3 > 0 \qquad \lambda = 6 > 0$

Suppose A is a 4×7 matrix whose reduced row echelon form is

$$\text{rref}(A) = \begin{bmatrix} \boxed{1} & 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Which of the following statements are true?

F (I) $\text{rank}(A) = 4$. → count pivots (3)   F

F (II) The dimension of the *Column Space* of A must be 4. → pivot columns in row echelon correspond to linearly indep columns in original matrix

↳ # of linearly indep vectors

T (III) The dimension of the *Solution Space* for $Ax = 0$ must be 4.

↳ null space : dimension = # of free variables in $A\vec{x} = \vec{0}$

= # of non pivot cols

= (# cols) − (# pivots)

T (IV) A basis for the *Row Space* of A is

$$\{\underbrace{[1\ 0\ 1\ 0\ -1\ 1\ 0],\ [0\ 0\ 0\ 1\ 2\ -3\ 0],\ [0\ 0\ 0\ 0\ 0\ 0\ 1]}_{\text{pivot rows in (reduced) row echelon}}\}.$$

A. Only (IV) is true

B. Only (II) and (III) are true

C. Only (II), (III) and (IV) are true

D. Only (III) and (IV) are true

E. All four statements are true

Let $x(t)$ satisfy the initial value problem

$$x'' + 4x' + 5x = 0,$$
$$x(0) = 1, \quad x'(0) = -2.$$

Then $x(2\pi)$, that is $x(t)$ evaluated at $t = 2\pi$, is equal to

A. $x(2\pi) = e^{4\pi}$

B. $x(2\pi) = e^{-4\pi}$

C. $x(2\pi) = e^{-2\pi}$

D. $x(2\pi) = e^{6\pi}$

E. $x(2\pi) = e^{-6\pi}$

$r^2 + 4r + 5 = 0$

$\cancel{(r \quad)(r \quad) = 0}$

$r = \dfrac{-4 \pm \sqrt{16-20}}{2} = \dfrac{-4 \pm 2i}{2} = -2 \pm i$

$x(t) = c_1 e^{-2t}\cos(t) + c_2 e^{-2t}\sin(t)$

$x(0) = 1 = c_1$

$x'(t) = -c_1 e^{-2t}\sin(t) - 2c_1 e^{-2t}\cos(t)$
$\quad + c_2 e^{-2t}\cos(t) - 2c_2 e^{-2t}\sin(t)$

$x'(0) = -2 = -2c_1 + c_2$

$-2 = -2 + c_2 \qquad c_2 = 0$

$x(t) = e^{-2t}\cos(t) \qquad x(2\pi) = e^{-4\pi}$

3. Find the explicit solution of the initial value problem

$$y' = \frac{xy^2}{x^2+1} \quad , \quad y(0) = 3.$$

separable?

A. $\frac{1}{2}(6 + \ln(1 + x^2))$

B. $\dfrac{6}{2 - 3\ln(1 + x^2)}$

C. $\dfrac{6}{2 + 3\ln(1 + x^2)}$

D. $\frac{1}{2}(6 - 3\ln(1 + x^2))$

E. $\frac{1}{3}(9 - 2\ln(1 + x^2))$

$$\frac{1}{y^2}\, y' = \frac{x}{x^2+1}$$

$$\int \frac{1}{y^2}\, dy = \int \frac{x}{x^2+1}\, dx$$

$$-\frac{1}{y} = \frac{1}{2}\ln(x^2+1) + C$$

$y(0)=3$

$$-\frac{1}{3} = \frac{1}{2}\ln(1) + C = C$$

$$-\frac{1}{y} = \frac{1}{2}\ln(x^2+1) - \frac{1}{3}$$

$$y = \frac{-1}{\frac{1}{2}\ln(x^2+1) - \frac{1}{3}} = \frac{6}{2 - 3\ln(x^2+1)}$$

9. Let A_{ij} be the cofactor of the element a_{ij} in the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ with $\det(A) = 5$. The value of the expression $a_{11}A_{11} + a_{12}A_{12} + a_{21}A_{21} + a_{22}A_{22}$ then is equal to

A. 0

B. 5

C. 10

D. 15

E. undetermined by the information given above

cofactor of element is the determinant of the matrix after crossing out the row and column the element is at and w/ correct sign

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ cofactor of 6 is $-\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix}$

here, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ A_{11} is cofactor of a_{11} $A_{11} = a_{22}$

$A_{12} = -a_{21}$ $A_{21} = -a_{12}$ $A_{22} = a_{11}$

$a_{11}A_{11} + a_{12}A_{12} + a_{21}A_{21} + a_{22}A_{22} = a_{11}a_{22} - a_{12}a_{21} + -a_{21}a_{12}$
$+ a_{22}a_{11}$

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = 5 = a_{11}a_{22} - a_{21}a_{12}$$

so $a_{11}A_{11} + a_{12}A_{12} + a_{21}A_{21} + a_{22}A_{22} = 5 + 5 = 10$