2.2 Equilibrium and Stability

population models from ~~last tim~~ last time

$$\frac{dy}{dt} = ky \qquad\qquad \frac{dy}{dt} = ay - by^2$$

these are called <u>autonomous differential equations</u>

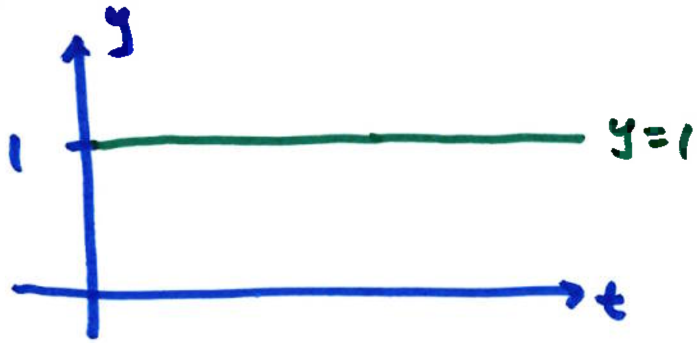
↳ does not depend explicitly on time (t)

$\frac{dy}{dt} = f(y)$ where $\frac{dy}{dt} = 0 \rightarrow y$ is called a <u>critical point</u>

for example, $\frac{dy}{dt} = y - 1$ then $y = 1$ is a critical point because if $y = 1$ then $\frac{dy}{dt} = 0$

the constant solution $y = c$ where c is a critical point is called an <u>equilibrium solution</u>

in $\frac{dy}{dt} = y - 1$, the equilibrium solution is $y = 1$



when population is 1, there is no change

if $y(0) = 1$, then $y = 1$ for all t

but what if $y(0)$ is greater than or less than 1?

what happens as time goes on?

one way to answer this is to solve the equation

$$\frac{dy}{dt} = y - 1 \qquad y(0) = y_0$$

is separable

$$\frac{1}{y-1}\, dy = dt$$

$\ln|y-1| = t + C$

$y - 1 = Ce^{t}$

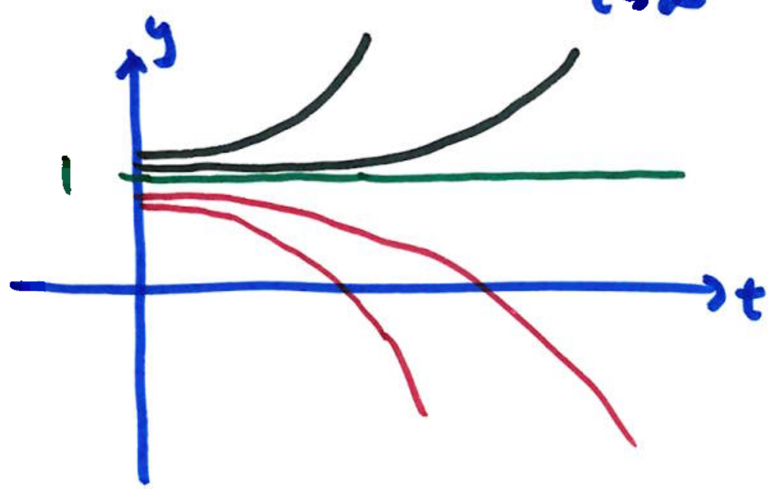
$y = 1 + Ce^{t}$ $\qquad y(0) = y_0$

$y_0 = 1 + C \qquad C = y_0 - 1 \qquad$ so $\boxed{y = 1 + (y_0 - 1)e^{t}}$

equilibrium: $y_0 = 1 \rightarrow y = 1$ for all t

if $y_0 > 1$, then $y_0 - 1 > 0$ so $\lim\limits_{t\to\infty} y = \infty$

if $0 < y_0 < 1$, then $\lim\limits_{t\to\infty} y = -\infty$

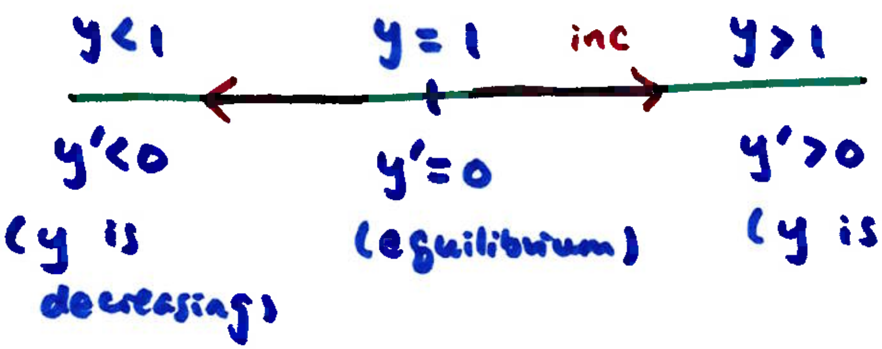


this is an <u>unstable equilibrium</u>

any perturbation from the equilibrium makes solution leave the equilibrium we can use the phase diagram to classify the equilibrium without solving the equation

$$\frac{dy}{dt} = y - 1$$

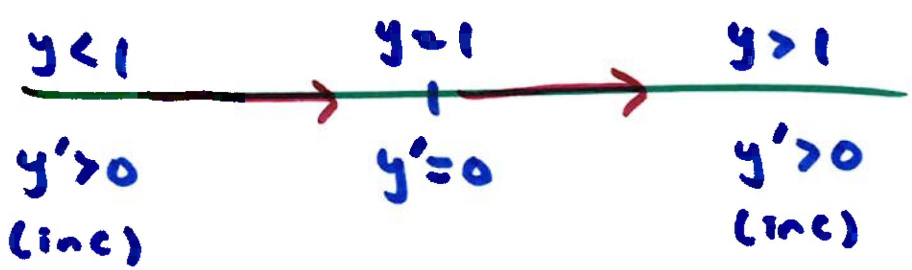
$y = 1$ is an equilibrium



So we see $y=1$ is an unstable equilibrium

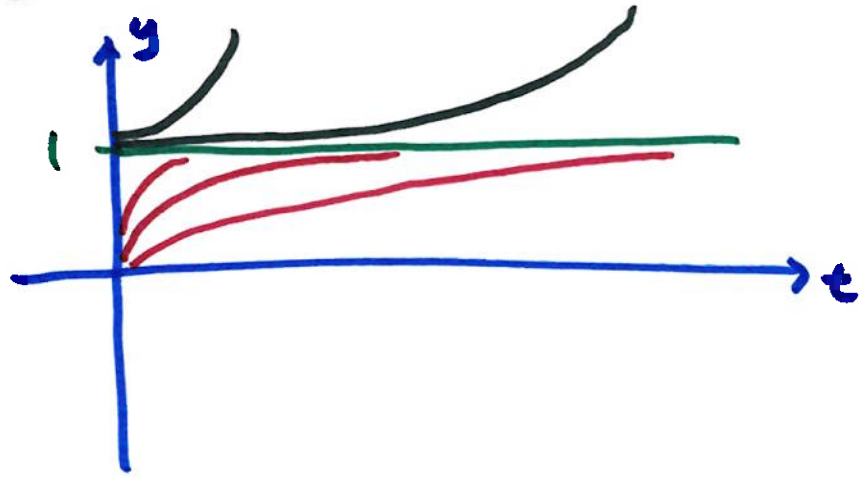
$$\frac{dy}{dt} = (y-1)^2$$

equilibrium: $y = 1$

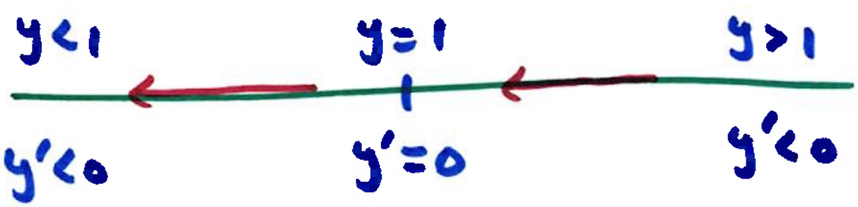


this is a semi-stable equilibrium

here, stable to the left of 1 unstable to the right qualitative sketch of solution curves



if $\dfrac{dy}{dt} = -(y-1)^2$



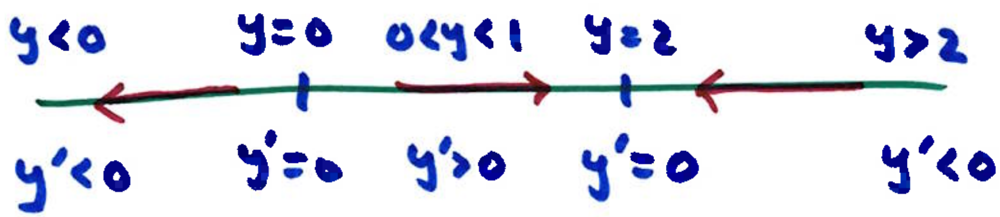
semi-stable

logistic growth: $\frac{dy}{dt} = ky(M-y)$ M: carrying capacity

for example, $\frac{dy}{dt} = y(2-y)$

two critical points: $y=0$, $y=2$

two equilibrium solutions: $y=0$, $y=2$



$y=0$ is unstable

$y=2$ is stable or asymptotically stable qualitative sketch of solution curves