2.4 Numerical Approx. : Euler's Method

we know how to solve many types of 1st-order: linear
separable
homogeneous
Bernoulli

and certain second-order by subs.

but there are equations that cannot be solved analytically

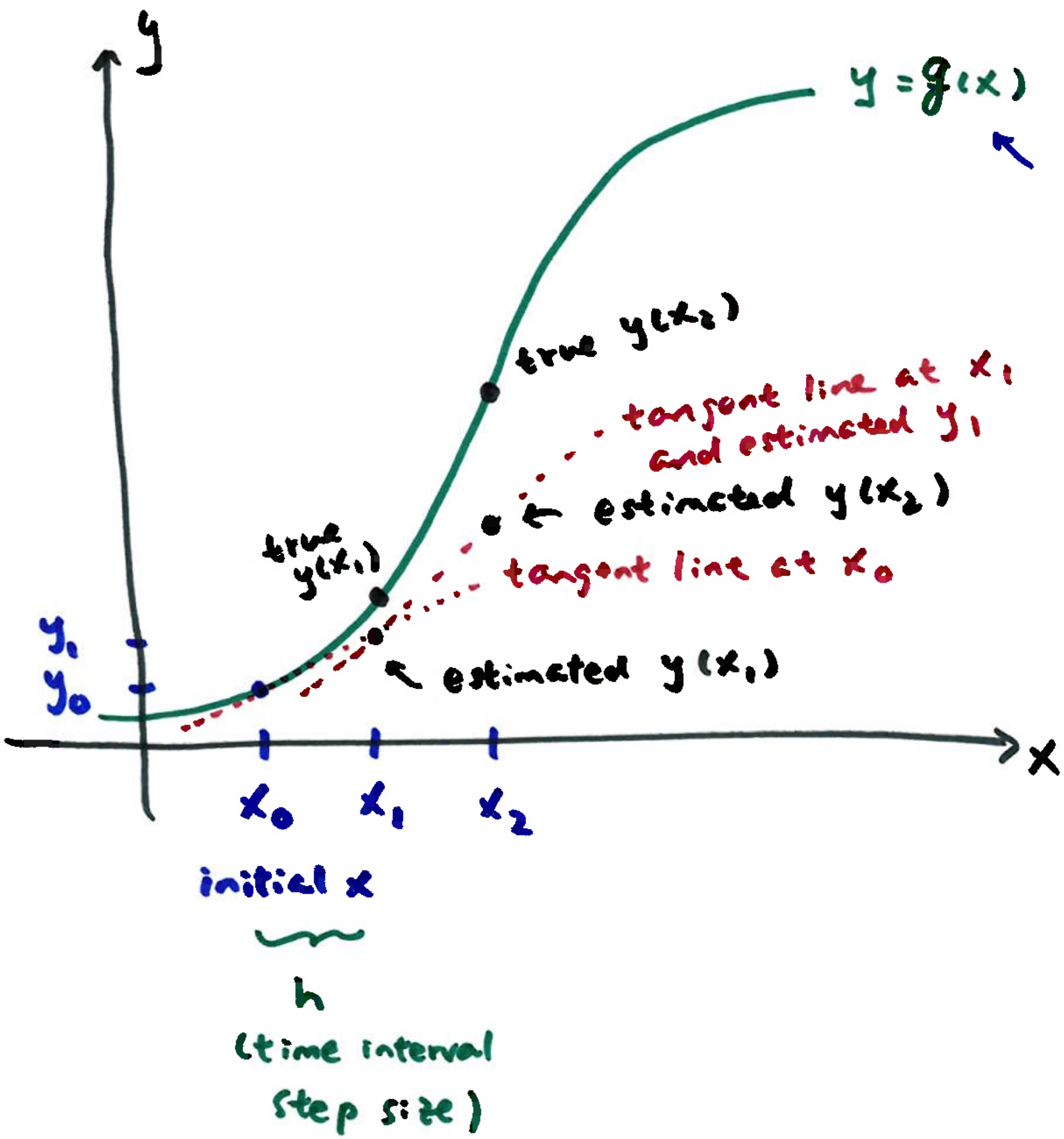
for example $y' = \dfrac{x + 2y^2 - 10}{\tan(xy)}$

Sometimes we can solve analytically but we may not want to for practical reasons

we can use numerical methods to approximate solutions

(computers use numerical methods)

Euler's Method (tangent line method)



$y' = f(x, y)$

Solution: $y = ?$

solution that we want

known: y at x_0

$y(x_0) = y_0$

want: y at x_2

$y(x_2) = ?$

we do <u>NOT</u> know the green curve but we know its slope

$y' = f(x, y)$

then continue until we reach target x

Algorithm outline

x_0 : given   want $y(x)$ for specified x

y_0 : given

Decide step size h

$\boxed{x_1 = x_0 + h}$ new x

tangent line at x_0: $y - y_0 = f'(x_0, y_0)(x - x_0)$

then estimate y_1 : $\boxed{y_1 = y_0 + f'(x_0, y_0)\, h}$

$x_2 = x_1 + h$

$y_2 = y_1 + f'(x_1, y_1)\, h$

⋮ repeat

$\boxed{y_{n+1} = y_n + f'(x_n, y_n)\, h}$ $\boxed{x_{n+1} = x_n + h}$

repeat until we reach x_n = target x

<u>example</u> $y' = 2y - 3x \quad y(0) = 1$

estimate $y(0.5)$ using step size of $h = 0.25$

$x_0 = 0$

$y_0 = 1$

$x_1 = x_0 + h = 0 + 0.25 = 0.25$

$y_1 = y_0 + f'(x_0, y_0)h$

$$= 1 + \underbrace{\Big[2\overset{\text{old } y}{(1)} - 3\overset{\text{old } x}{(0)}\Big]}_{\text{right side of DE}}(0.25) = 1.5$$

$x_2 = x_1 + h$

$= 0.25 + 0.25 = 0.5$ target x so this is the final step

$y_2 = y_1 + f'(x_1, y_1)h$

$$= 1.5 + \Big[2\overset{y_1}{(1.5)} - 3\overset{x_1}{(0.25)}\Big](0.25) = \boxed{2.0625}$$

estimate of $y(0.5)$
from $y' = 2y - 3x \quad y(0) = 1$

how good is the estimate?

in this example, $y' = 2y - 3x$ $y(0)=1$ can be solved analytically

so we can compare (generally is NOT the case)

$$y' - 2y = -3x \qquad I = e^{\int -2\,dx} = e^{-2x}$$

$$e^{-2x}(y' - 2y) = -3xe^{-2x}$$

$$\frac{d}{dx}(e^{-2x}y) = -3xe^{-2x}$$

$$e^{-2x}y = \int -3xe^{-2x}\,dx \quad \text{(by parts)}$$

$$\vdots$$

$$y = \frac{3}{4}(2x+1) + \frac{1}{4}e^{2x}$$

Euler method estimate : $y(0.5) = 2.0625$

True value from analytical solution: $y(0.5) = 2.1796$

To improve, use more steps (small step size h)

if we had used $h = 0.01$ here, the estimate would

be $y(0.5) = 2.1729$ (50 steps)

in general, we don't know the true solution y

(otherwise why bother w/ Euler?)

so how to tell the estimate $y(x)$ is good?

→ if estimate is good, further refinement of h would result in very small change in estimated y

("the solution has converged")

for example, $y' = 2y - 3x$ $\quad y(0) = 1$ $\quad$ estimate $y(1)$

$h = 0.5$ $\quad y(1) = 3.25$

$h = 0.05$ $\quad y(1) = 3.932$

big change, keep shrinking h

$h = 0.01$ $\quad y(1) = 4.061$

still changing, keep shrinking

$h = 0.001$ $\quad y(1) = 4.094$

$h = 0.0005$ $\quad y(1) = 4.095$

seems to be settling down ("converging") so further refinement of h is not worth the effort

so, the estimate of $y(1) = 4.095$ is good enough