3.1 Introduction to Linear Systems

in 2D, $ax + by = c$ a, b, c constants

is a line

a system in 2D:

$$\left.\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}\right\}$$

2 unknowns: x, y

2 equations

Solution: all x, y that satisfy both equations

for example,

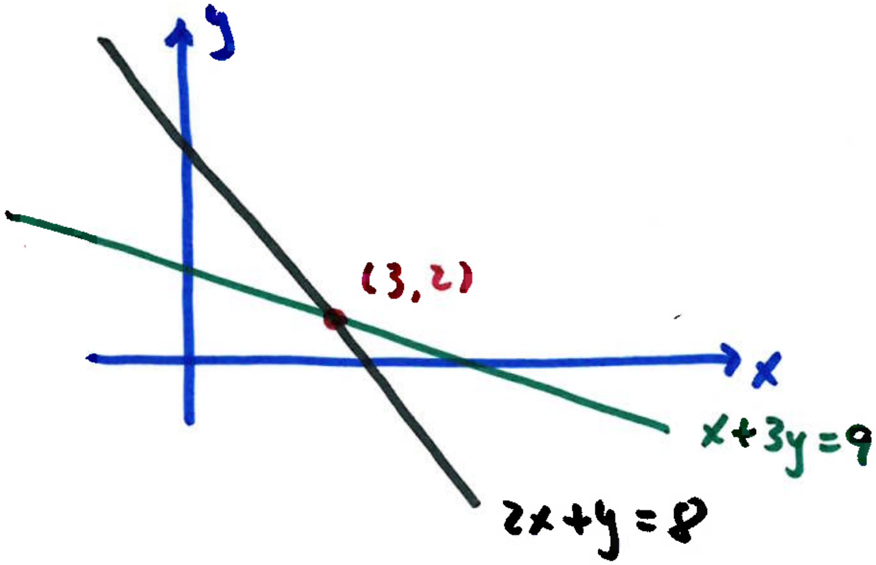
$$x + 3y = 9$$
$$2x + y = 8$$

has solution $x = 3$, $y = 2$

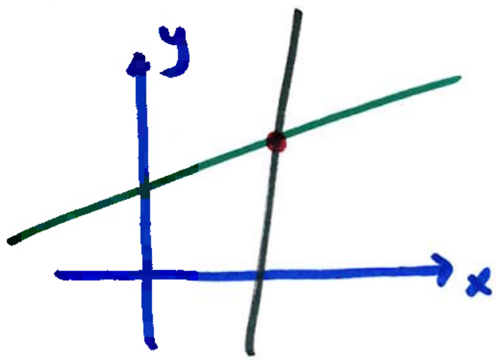
verify: $3 + 3(2) = 9$ is true (eq. 1)

$2(3) + 2 = 8$ is true (eq. 2)

Solution (x, y) is both lines

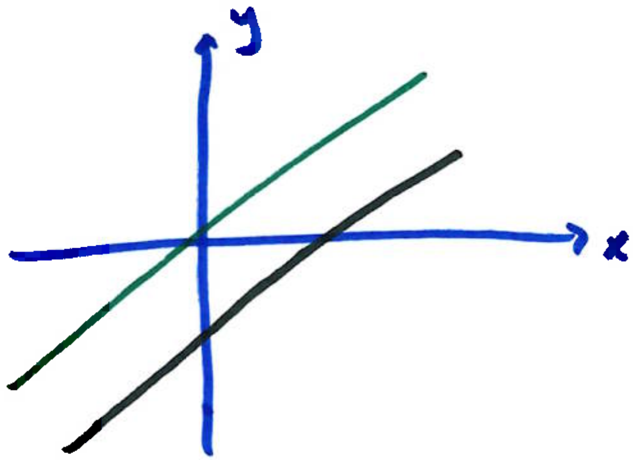


In general, in 2D, there are three possibilities



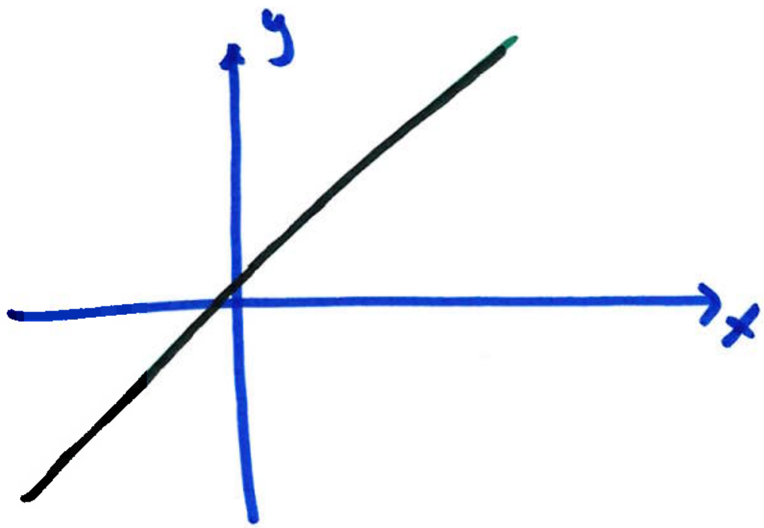
two lines intersect at one point

→ one solution (x, y)



two parallel lines

→ never intersect, so no solution



two lines on top of each other

→ infinitely-many common points

infinitely-many solutions

2D system: one solution
inf. many solutions } system is <u>consistent</u>
no solution → system is <u>inconsistent</u>

One method to solve a linear system is elimination

↳ multiply one equation by some number, add to the other such that one variable is eliminated, solve for one, find the other by back substitution

example $x + 3y = 9$ — ①

$2x + y = 8$ — ②

multiply ① by -2, add to ②
to eliminate x

$-2x - 6y = -18$ $\quad -2$①

$-2$① $+$②

$-5y = -10$ so, $\boxed{y = 2}$

back sub into ① or ②

①: $x + 3(2) = 9$

$\boxed{x = 3}$

example

$$x + 2y = 4 \quad - ①$$
$$2x + 4y = 9 \quad - ②$$

$-2① + ②$

$0 = 1$ nonsense ! no (x, y) that can satisfy BOTH equations

no solution (lines parallel)

inconsistent system

example

$$x + 2y = 4 \quad - ①$$
$$2x + 4y = 8 \quad - ②$$

$-2① + ②$

$0 = 0$ true any (x, y) that satisfies one equation satisfies the 2nd one, too

two same lines

infinitely-many solutions

$x + 2y = 4$ choose one : $x = t$

$2y = 4 - t$

$y = 2 - \frac{1}{2}t$ solution: $(t, 2 - \frac{1}{2}t)$ t real

or choose $y = t$

then $x = 4 - 2t$ solution $(4-2t, t)$

3D: $$\left.\begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned}\right\} \text{ 3 planes}$$

Solution: (x, y, z) that lies on all 3 planes

one possiblity : $\left.\begin{array}{l}\text{one point} \\ \text{one line}\end{array}\right\}$ consistent

no common line / point inconsistent

elimination works the same way

example

$$x + 5y + z = 2 \quad - ①$$

$$2x + y - 2z = 1 \quad - ②$$

$$x + 7y + 2z = 3 \quad - ③$$

$-2①+②$ $\quad -9y - 4z = -3 \quad - ④$

$-①+③$ $\quad 2y + z = 1 \quad - ⑤$

solve 2D sys for y, z

$4⑤+④$ $\quad -y = 1 \rightarrow \boxed{y = -1}$

from ⑤ $\quad z = 1 - 2y = 1 - 2(-1) = 3 \quad \boxed{z = 3}$

from ① $\quad x = 2 - 5y - z$

$$= 2 - 5(-1) - (3)$$

$$= 4$$

$$\boxed{x = 4}$$

we use elimination to find constants of integration in higher-order differential equations

example $y'' + 4y' - 21y = 0 \quad y(0) = 35, \; y'(0) = -45$

solution $y(x) = Ae^{3x} + Be^{-7x}$

Find A, B.

$y(0) = 35 \longrightarrow 35 = A + B$ — ①

need $y'(x)$ to use $y'(0)$

$y'(x) = 3Ae^{3x} - 7Be^{-7x}$

$y'(0) = -45 \longrightarrow -45 = 3A - 7B$ — ②

solve ①, ② by elimination

$-3① + ②$ $\quad -150 = -10B \quad \boxed{B = 15}$

from ① $\quad A = 35 - B \quad \boxed{A = 20}$