3.2 Matrices and Gaussian Elimination

from last time:

$$\begin{aligned} x_1 + 3x_2 &= 9 \\ 2x_1 + x_2 &= 8 \end{aligned}$$

we solved by elimination, we got $x_1 = 3$, $x_2 = 2$

we can represent the linear system above this way

$$\begin{array}{ccc} x_1 & x_2 & \text{right} \\ \end{array}$$
$$\begin{bmatrix} 1 & 3 & 9 \\ 2 & 1 & 8 \end{bmatrix}$$

this is a matrix – a rectangular array of numbers

this is the coefficient matrix representing the system above (augmented matrix because last column is the right side of eqs)

this matrix has two rows and three columns

→ this is a two by three matrix or 2×3

elimination method from last time are accomplished by

<u>elementary row operations</u>

1. swap any two rows
2. add constant multiple of one row to another
3. multiply any row by nonzero constant

$$\begin{bmatrix} 1 & 3 & 9 \\ 2 & 1 & 8 \end{bmatrix} \xrightarrow{(-2)R_1 + R_2} \begin{bmatrix} 1 & 3 & 9 \\ 0 & -5 & -10 \end{bmatrix}$$

"stairs"

$$\xrightarrow{(-\frac{1}{5})R_2} \begin{bmatrix} 1 & 3 & 9 \\ 0 & 1 & 2 \end{bmatrix}$$

this is in the <u>echelon form</u>

echelon form: a row with all zeros (if any) is at the bottom

the <u>leading entry</u> (left most element in a row) of every nonzero row, called the <u>pivot</u>, is to the right and below the pivot of the row above.

$$\begin{array}{ccc} \text{pivot} & & \\ \end{array}$$

$$\begin{bmatrix} \boxed{1} & 3 & 9 \\ 0 & \boxed{1} & 2 \end{bmatrix}$$

$$\begin{array}{ccc} & \text{pivot} & \\ \end{array}$$

solve from echelon form is easy

$R_2: \; 0 \cdot x_1 + 1 \cdot x_2 = 2 \longrightarrow \boxed{x_2 = 2}$

$R_1: \; 1 \cdot x_1 + 3 \cdot x_2 = 9$

$$x_1 = 9 - 3x_2 = 9 - 3(2) = 3 \qquad \boxed{x_1 = 3}$$

the process using row operations to reduce a matrix to solve the system is called <u>Gaussian Elimination</u>

or <u>row reduction</u>

all the matrices that result from row ops are said to be <u>row equivalent</u> (not "equal") example

$$x_2 + 4x_3 = -3$$
$$x_1 + 3x_2 + 6x_3 = 4$$
$$2x_1 + 5x_2 + 8x_3 = 5$$

matrix form $\begin{bmatrix} 0 & 1 & 4 & -3 \\ 1 & 3 & 6 & 4 \\ 2 & 5 & 8 & 5 \end{bmatrix}$ 3x4 matrix

goal: echelon form: $\begin{bmatrix} \overset{\text{non zero}}{x} & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{bmatrix}$

generally we want non zero top most left element

$$\xrightarrow{\text{swap}(R_1, R_2)} \begin{bmatrix} 1 & 3 & 6 & 4 \\ 0 & 1 & 4 & -3 \\ 2 & 5 & 8 & 5 \end{bmatrix}$$

pivot of col 1

$$\xrightarrow{(-2)R_1 + R_3} \begin{bmatrix} \boxed{1} & 3 & 6 & 4 \\ 0 & \boxed{1} & 4 & -3 \\ 0 & -1 & -4 & -3 \end{bmatrix}$$

pivot col 2

$$\xrightarrow{R_2 + R_3} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \begin{bmatrix} 1 & 3 & 6 & 4 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & -6 \end{bmatrix} \end{array}$$

echelon form. now solve

R_3 : $0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = -6 \rightarrow 0 = -6$

no solution

<u>example</u>

$$x_1 - x_2 + x_3 = 7$$
$$3x_1 + 2x_2 - 12x_3 = 11$$
$$4x_1 + x_2 - 11x_3 = 18$$

$$\begin{bmatrix} 1 & -1 & 1 & 7 \\ 3 & 2 & -12 & 11 \\ 4 & 1 & -11 & 18 \end{bmatrix}$$

$$\xrightarrow{(-3)R_1 + R_2} \begin{bmatrix} 1 & -1 & 1 & 7 \\ 0 & 5 & -15 & -10 \\ 4 & 1 & -11 & 18 \end{bmatrix}$$

$$\xrightarrow{(-4)R_1 + R_3} \begin{bmatrix} 1 & -1 & 1 & 7 \\ 0 & 5 & -15 & -10 \\ 0 & 5 & -15 & -10 \end{bmatrix}$$

$$\xrightarrow{(-1)R_2+R_3} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{cccc} \boxed{1}^{\text{pivot}} & -1 & 1 & 7 \\ 0 & \boxed{5}^{\text{pivot}} & -15 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{echelon form}$$

$R_3: \quad 0 \cdot x_1 + 0 \cdot x_2 + 0 \cdot x_3 = 0 \rightarrow \boxed{0 = 0}$

zero row : at least one variable is free (arbitrary)

# of zero rows = # free variables

$R_2: \quad 0 \cdot x_1 + 5 \cdot x_2 + -15 \cdot x_3 = -10$

$$5x_2 - 15x_3 = -10$$

$$x_2 - 3x_3 = -2$$

one of the variables (x_1, x_2, x_3) is free

third column does not have a pivot so normally we choose its variable (x_3) to be free

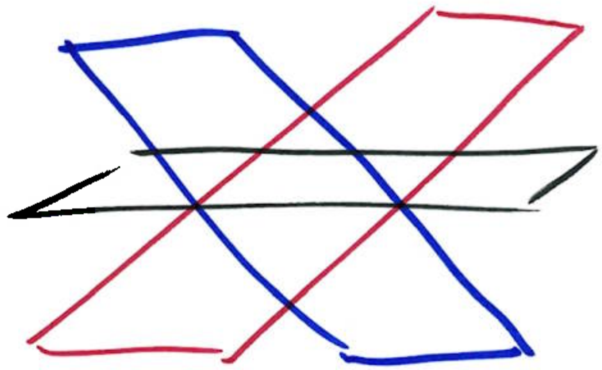
$$x_3 = t$$

$R_2: \quad x_2 - 3x_3 = -2 \rightarrow x_2 = 3x_3 - 2 = 3t - 2$

$R_1: \quad x_1 - x_2 + x_3 = 7 \rightarrow x_1 = 7 + x_2 - x_3 = 7 + 3t - 2 - t$

$$x_1 = 5 + 2t$$

So the three planes intersect at a line



side view