3.3 Reduced Row Echelon Form

row echelon: zero rows at bottom

pivot of each nonzero row is below and to the right of the pivot of the row above

reduced echelon: in addition to the above, we want

each pivot to be 1

each pivot is the only nonzero element in its column

example

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

make this go away

$$\xrightarrow{(-3)R_1+R_2} \begin{bmatrix} 1 & \boxed{2} \\ 0 & \boxed{-2} \end{bmatrix} \quad \text{echelon}$$

make this a 1

pivot is 1

only nonzero in its column

$$\xrightarrow{R_2+R_1} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \xrightarrow{(-\frac{1}{2})R_2} \begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{bmatrix}$$

echelon is NOT unique

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow{swap(R_1, R_2)} \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \xrightarrow{(-\frac{1}{3})R_1+R_2} \begin{bmatrix} 3 & 4 \\ 0 & \frac{2}{3} \end{bmatrix}$$

echelon

reduce it more:

$$\xrightarrow{(\frac{1}{3})R_1} \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & \frac{2}{3} \end{bmatrix} \xrightarrow{(-2)R_2+R_1} \begin{bmatrix} 1 & 0 \\ 0 & \frac{2}{3} \end{bmatrix}$$

$$\xrightarrow{(\frac{3}{2})R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

reduce row echelon form

(rref) is unique

while echelon form is NOT

example

$$\begin{aligned} x_1 + 3x_2 + 2x_3 &= 12 \\ 2x_1 + 5x_2 + 2x_3 &= 12 \\ 2x_1 + 7x_2 + 7x_3 &= 43 \end{aligned}$$

$$\begin{bmatrix} 1 & 3 & 2 & 12 \\ 2 & 5 & 2 & 12 \\ 2 & 7 & 7 & 43 \end{bmatrix}$$

make it rref

Gauss-Jordan Elimination

pivot = 1

only nonzero in col 1

$$\xrightarrow[(-2)R_1 + R_3]{(-2)R_1 + R_2} \begin{bmatrix} \boxed{1} & 3 & 2 & 12 \\ 0 & -1 & -2 & -12 \\ 0 & 1 & 3 & 19 \end{bmatrix}$$

$$\xrightarrow{R_2 + R_3} \begin{bmatrix} 1 & 3 & 2 & 12 \\ 0 & -1 & -2 & -12 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

$$\xrightarrow{(3)R_2+R_1} \begin{bmatrix} 1 & 0 & -4 & -24 \\ 0 & -1 & -2 & -12 \\ 0 & 0 & 1 & 7 \end{bmatrix} \xrightarrow{(-1)R_2} \begin{bmatrix} \boxed{1} & 0 & -4 & -24 \\ 0 & \boxed{1} & 2 & 12 \\ 0 & 0 & \boxed{1} & 7 \end{bmatrix}$$

want to be only nonzero in col 3

because there is no room since each pivot is to the right and below of prev. one

no pivot here don't care what it looks like

$$\xrightarrow[(4)R_3+R_1]{(-2)R_3+R_2} \begin{bmatrix} \boxed{1} & 0 & 0 & 4 \\ 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{1} & 7 \end{bmatrix}$$

this <u>is</u> rref

R_3 : $0x_1 + 0x_2 + x_3 = 7 \rightarrow x_3 = 7$

R_2 : $x_2 = -2$

R_1 : $x_1 = 4$

example

$$\begin{aligned} x_1 + 3x_2 - 15x_3 + 7x_4 &= 0 \\ x_1 + 4x_2 - 19x_3 + 10x_4 &= 0 \\ 2x_1 + 5x_2 - 26x_3 + 11x_4 &= 0 \end{aligned}$$

right side all zeros

→ homogeneous system

↓

has NOTHING to do with $v = \frac{y}{x}$

$$\begin{bmatrix} 1 & 3 & -15 & 7 & 0 \\ 1 & 4 & -19 & 10 & 0 \\ 2 & 5 & -26 & 11 & 0 \end{bmatrix}$$

$$\xrightarrow[(-2)R_1 + R_3]{(-1)R_1 + R_2} \begin{bmatrix} 1 & 3 & -15 & 7 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & -1 & 4 & -3 & 0 \end{bmatrix}$$

$$\xrightarrow[(-3)R_2 + R_1]{R_2 + R_3} \begin{bmatrix} \boxed{1} & 0 & -3 & -2 & 0 \\ 0 & \boxed{1} & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{rref}$$

$0 = 0$ → presence of free variables (arbitrary value)

$$\#\text{ free variables} = (\#\text{ variables}) - (\#\text{ pivots})$$
$$= 4 - 2 = 2$$

pivots in columns 1 and 2 but not column 3 and 4

choose x_3 and x_4 to be free

$x_3 = s \qquad x_4 = t \qquad s, t$ real numbers

$R_2: \quad x_2 - 4x_3 + 3x_4 = 0 \quad \rightarrow \quad x_2 = 4s - 3t$

$R_1: \quad x_1 - 3x_3 - 2x_4 = 0 \quad \rightarrow \quad x_1 = 3s + 2t$

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zero row $\rightarrow$ free variables $\rightarrow$ infinitely-many solutions

$\rightarrow$ # pivots < # variables

unique solution: # pivots = # variables

EXCEPT if a pivot is in the LAST column (right side of equations and not associated with any variable) example A certain system has this rref of its augmented coefficient matrix

$$\begin{array}{cccc} x_1 & x_2 & x_3 & \text{right side number} \\ \end{array}$$

$$\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

R_3 : $0 = 1$ → no solution

all homogeneous systems (right side all zeros) has the trivial solution (all variables are zero)

earlier example

$$\left.\begin{array}{l} x_3 = s \quad x_4 = t \\ x_2 = 4s - 3t \\ x_1 = 3s + 2t \end{array}\right\} \quad \begin{array}{l} \text{if } s = t = 0 \\ x_1 = x_2 = x_3 = x_4 = 0 \end{array}$$