3.4 Matrix Operations

many operations we use with scalars carry over to matrices

BUT some operations w/ matrices are very different

notation: usually use capital letters for matrices e.g. A

in print (books) usually bold faced capital letters

$A = [a_{ij}]$ A has element a_{ij} in the i^{th} ~~column~~ row and j^{th} column

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Addition: if two matrices of the same size, then addition is element-by-element

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & -2 \\ -7 & 10 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2×2 2×2 3×1

$$A+B = \begin{bmatrix} 1+-3 & 2+-2 \\ 3+-7 & 4+10 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ -4 & 14 \end{bmatrix}$$

$A+C$ is NOT possible (not the same size)

<u>Multiplication by a scalar</u> : multiply to every element

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3\cdot 1 & 3\cdot 2 \\ 3\cdot 3 & 3\cdot 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

$$-3B = \begin{bmatrix} -3\cdot 1 & -3\cdot 2 & -3\cdot 3 & -3\cdot 4 \end{bmatrix} = \begin{bmatrix} -3 & -6 & -9 & -12 \end{bmatrix}$$

<u>Subtraction</u> $A-B = A+(-1)B$ $\quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

$$A-B = A+(-1)B$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} -4 & -3 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 1 & 3 \end{bmatrix}$$

a matrix of one column or one row is called a <u>vector</u>

$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is a column vector

$B = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$ is a row vector

vectors are often denote by lowercase letter with arrow over it or boldfaced

$\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

systems of vector or matrix equations

$$\begin{aligned} 2x_1 + 3x_2 - 4x_3 &= 1 \\ -x_2 + 5x_3 &= 2 \\ 3x_1 + 4x_2 \quad &= 3 \end{aligned}$$

as a vector/matrix eg: $x_1 \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

two ways to interpret solving this system:

1. Intersection of planes
2. choosing the weights of $\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 0 \end{bmatrix}$ such that the linear combination of them is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Multiplication of matrices

AB : # of columns of first matrix (A)
= # of rows of second (B)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

$2 \times \boxed{3}$ cols $\qquad \boxed{3} \times 1$ rows

so AB is allowed while BA is NOT

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

$$\underline{\underline{2}} \times \underbrace{\boxed{3 \quad\quad\quad 3}}_{\text{match}} \times \underline{\underline{1}}$$

size of resulting matrix AB → 2 × 1 matrix

$$\begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ \boxed{0 \quad -1 \quad -2} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 4 \\ 5 \end{matrix}} \end{bmatrix} = \begin{bmatrix} \boxed{24} \\ \boxed{-14} \end{bmatrix}$$

2 × 1

multiply row 1 of 1st by col 1 of 2nd
element by element and add

$1 \cdot 1 + 2 \cdot 4 + 3 \cdot 5 = 24$

then repeat

$0 \cdot 1 + -1 \cdot 4 + -2 \cdot 5 = -14$

<u>example</u>

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 0 & 3 \\ 1 & -5 & 2 \end{bmatrix}$$

$$2 \times 2 \qquad\qquad 2 \times 3$$

is AB permissible? yes

BA? no

$$AB = \underset{\boxed{2}\times 2}{\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}} \underset{2 \times \boxed{3}}{\begin{bmatrix} -4 & 0 & 3 \\ 1 & -5 & 2 \end{bmatrix}} = \underset{2\times 3}{C}$$

row 1 of A times col 1 B

R1 of A times C1 of B

R2 of A times C3 of B

$$= \begin{bmatrix} 1\cdot -4 + 1\cdot 1 & 1\cdot 0 + 1\cdot -5 & 1\cdot 3 + 1\cdot 2 \\ 2\cdot -4 + 1\cdot 1 & 2\cdot 0 + 1\cdot -5 & 2\cdot 3 + 1\cdot 2 \end{bmatrix} = \begin{bmatrix} -3 & -5 & 5 \\ -7 & -5 & 8 \end{bmatrix}$$

example $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 9 \\ -5 & -5 \end{bmatrix}$

find AB and BA (if possible)

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 9 \\ -5 & -5 \end{bmatrix} = \begin{bmatrix} -7 & 13 \\ -6 & 4 \end{bmatrix}$$

$$BA = \begin{bmatrix} -1 & 9 \\ -5 & -5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ -15 & -10 \end{bmatrix}$$

~~if~~ in general, $AB \neq BA$ even if possible to do both

there is no such thing as matrix division matrix equation $A\vec{x} = \vec{b}$

$$x_1 - x_3 + 6x_4 + 5x_5 = 0$$

$$x_2 + 9x_3 - 2x_4 + 4x_5 = 0$$

can be expressed as

$$\underbrace{\begin{bmatrix} 1 & 0 & -1 & 6 & 5 \\ 0 & 1 & 9 & -2 & 4 \end{bmatrix}}_{A \text{ : matrix of coefficients}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{\vec{x} \text{ : variables}} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{\vec{b} \text{ : right side}}$$

Solution is easy because A is echolon (reduced)

2 pivots, 5 variables $\rightarrow$ 3 variables are free : x_3, x_4, x_5

$x_5 = t$, $x_4 = s$, $x_3 = r$

R2: $x_2 = -9x_3 + 2x_4 - 4x_5 = -9r + 2s - 4t$

R1: $x_1 = r - 6s - 5t$

write out $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$