3.5 Inverses of Matrices

scalar: "1" is a number such that $1 \cdot a = a \cdot 1 = a$

and the scalar inverse is a^{-1} such that $a^{-1}a = aa^{-1} = 1$

matrix: I is the <u>identity matrix</u> ~~such~~ such that $IA = AI = A$

and matrix inverse A^{-1} is such that $A^{-1}A = AA^{-1} = I$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2×2

ones on <u>main diagonal</u>, zeros everywhere else

$$n \times n: \quad I = \begin{bmatrix} 1 & \cdots & 0 \\ 0 & \ddots & 0 \\ \vdots & & \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad AI = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\underbrace{A^{-1}A = AA^{-1}}= I$$

implies A has to be square matrix to have an inverse

not every matrix has an inverse (just like 0 (scalar) doesn't have an inverse)

if A^{-1} exists $\rightarrow$ A is invertible

if A^{-1} doesn't exist $\rightarrow$ A is singular

$$\boxed{\text{if } A \text{ is } 2\times 2,\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then } A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

$ad-bc$ is the determinant of A

A^{-1} exists only if $\det(A) \neq 0$

<u>example</u> $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$A^{-1} = \frac{1}{1\cdot 4 - 2\cdot 3}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

check: $AA^{-1} = I$ AND $A^{-1}A = I$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

inverse can be used to solve matrix equations

scalar: $ax = b \rightarrow x = a^{-1}b$ because $\underbrace{a^{-1}a}_{1}\, x = a^{-1}b$

matrix: $A\vec{x} = \vec{b} \rightarrow \underbrace{A^{-1}A}_{I}\,\vec{x} = A^{-1}\vec{b} = \vec{x} = A^{-1}\vec{b}$

example

$$x_1 + 2x_2 = 5$$
$$3x_1 + 4x_2 = 6$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 5 \\ 6 \end{bmatrix}}_{\vec{b}}$$

solved by $\vec{x} = A^{-1}\vec{b}$ $\qquad A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$ (last example)

$$\vec{x} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ \frac{9}{2} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

no formulas for A^{-1} for $n \times n$ where $n > 2$

but there is a general algorithm that works for all $n \times n$

make an augmented matrix w/ A on left and I on right

$$\left[A \;\vdots\; I \right]$$

for readability only

perform row operations until left half is I

then whatever is on the right is A^{-1}

$$\vdots$$

$$\left[I \;\vdots\; A^{-1} \right]$$

---

try on $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -2 & -3 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & -2 & -3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc:cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right]$$

$$A^{-1} = \boxed{\begin{matrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{matrix}}$$

<u>example</u>

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$ find A^{-1} if it exists

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{matrix}}_{A^{-1}}$$

<u>example</u>

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}$$

$$\left[\begin{array}{ccc:ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc:ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc:ccc} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & x & x & x \end{array}\right]$$

can't turn into I

so A is not invertible

why does $[A \,\vdots\, I] \xrightarrow{\text{ops}} [I \,\vdots\, A^{-1}]$ work ?

$$a x_1 + b x_2 = e$$
$$c x_1 + d x_2 = f$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

solution of x_1, x_2 is

$$\left[\begin{array}{cc:c} a & b & e \\ c & d & f \end{array}\right] \rightarrow \left[\begin{array}{c:c} I & \vec{x} \end{array}\right]$$

finding inverse is the same idea

$$AX = I \qquad X = A^{-1}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

same logic

$$\left[\begin{array}{cc:cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right]$$

$$\xrightarrow{\text{ops}} \left[\begin{array}{c:cc} I & \underbrace{\begin{matrix} e & f \\ g & h \end{matrix}}_{X = A^{-1}} \end{array}\right]$$