3.6 Determinants (part 1)

the determinant of matrix A, $\det A$ or $\det(A)$, tells us, among other things, whether A^{-1} exists.

if $\det A = 0$, then A^{-1} does not exist

if A is 2×2 : $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

for $n\times n$ $(n > 2)$, there are not convenient formulas like the above

the algorithm Cofactor Expansion allows us to write the determinant of a 3×3 matrix as sum of 2×2 determinants

(4×4 as sum of determinants of 3×3 and so on)

example

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 3 & 4 & 2 \\ 0 & 4 & -2 \end{bmatrix}$$

find det A

pick ANY row or column to expand

as an example, let's choose to expand along column 1

$(-1)^{i+j}$ i, j row, ~~row~~ col #

$$\det A = \underbrace{(2)\underbrace{\begin{vmatrix} 4 & 2 \\ 4 & -2 \end{vmatrix}}_{\substack{\text{det of matrix after} \\ \text{blocking out row 1, col 1} \\ \text{(minor of } a_{11})}}}_{\text{cofactor of } a_{11}} - (3)\begin{vmatrix} 0 & 4 \\ 4 & -2 \end{vmatrix} + (0)\begin{vmatrix} 0 & 4 \\ 4 & 2 \end{vmatrix}$$

a_{11} a_{21}

cofactor signs

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= (2)(-8-8) - (3)(0-16) + (0)(0-16) = \boxed{16}$$

try expanding along row 2

$$A = \begin{bmatrix} 2^{+} & 0^{-} & 4^{+} \\ 3^{-} & 4^{+} & 2^{-} \\ 0^{+} & 4^{-} & -2^{+} \end{bmatrix}$$

$$\det A = -(3)\begin{vmatrix} 0 & 4 \\ 4 & -2 \end{vmatrix} + (4)\begin{vmatrix} 2 & 4 \\ 0 & -2 \end{vmatrix} - (2)\begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix}$$

$$= (-3)(-16) + (4)(-4) + (-2)(8) = 16$$

Same determinant along any row/col $\rightarrow$ pick row/col w/ most zeros example

$$A = \begin{bmatrix} 8^{+} & 0^{-} & 0^{+} & 5^{-} \\ 5^{-} & 8^{+} & 3^{-} & -7^{+} \\ 2^{+} & 0^{-} & 0^{+} & 0^{-} \\ 7^{-} & 2^{+} & 1^{-} & 7^{+} \end{bmatrix}$$

row 3 has most zeros, so is a good row to expand along

$$\det A = \underbrace{(2)\begin{vmatrix} 0 & 0 & 5 \\ 8 & 3 & -7 \\ 2 & 1 & 7 \end{vmatrix}}_{\text{expand again row 1 looks good}} - (0)\cancel{\begin{vmatrix} 8 & 0 & 5 \\ 8 & 3 & -7 \\ 7 & 1 & 7 \end{vmatrix}} + (0)\cancel{\begin{vmatrix} \cdot & & \\ & \cdot & \\ & & \cdot \end{vmatrix}} - (0)\cancel{\begin{vmatrix} & \cdot & \\ & & \\ & \cdot & \end{vmatrix}}$$

$$= (2)(5)\begin{vmatrix} 8 & 3 \\ 2 & 1 \end{vmatrix}$$

$$= (2)\left[0 \begin{vmatrix} 3 & -7 \\ 1 & 7 \end{vmatrix} - 0 \begin{vmatrix} 0 & 5 \\ 2 & 7 \end{vmatrix} + 5 \begin{vmatrix} 8 & 3 \\ 2 & 1 \end{vmatrix} \right]$$

$$= (2)(5)(8-6) = 20$$

Application: Solving systems using <u>Cramer's Rule</u>

$$\cancel{a x_1 + b x_2 =}$$

$$\left.\begin{aligned} a_{11} x_1 + a_{12} x_2 &= b_1 \\ a_{21} x_1 + a_{22} x_2 &= b_2 \end{aligned}\right\} \quad \overbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

$\leftarrow$ det of A w/ col 1 replaced by $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

$\leftarrow$ det A

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}$$

$\leftarrow$ det A

<u>example</u> $\quad x_1 + 3x_2 = 9$ $\qquad$ we've seen this system before

$\qquad\qquad\quad 2x_1 + x_2 = 8$ $\qquad\qquad$ $(x_1 = 3, \quad x_2 = 2)$

$$x_1 = \frac{\begin{vmatrix} 9 & 3 \\ 8 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-15}{-5} = 3$$

$$x_2 = \frac{\begin{vmatrix} 1 & 9 \\ 2 & 8 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}} = \frac{-10}{-5} = 2$$

same idea for $n \times n$ for $n > 2$

find x_j $\rightarrow$ denominator is det A

$\rightarrow$ numerator determinant of A w/ j^{th} replaced by $\vec{b}$

<u>example</u>

$$x_1 + x_2 = 4$$
$$-4x_1 + 3x_3 = 0 \qquad \text{find } x_1$$
$$x_2 - 3x_3 = 3$$

rewrite: $\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ -4 & 0 & 3 \\ 0 & 1 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$

$$x_1 = \frac{\begin{vmatrix} 4 & 1 & 0 \\ 0 & 0 & 3 \\ 3 & 1 & -3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 0 \\ -4 & 0 & 3 \\ 0 & 1 & -3 \end{vmatrix}} = \cdots = \frac{1}{5}$$

first col replaced

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ the <u>transpose</u> of A is $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

interchange rows and cols

row1 → col 1, row 2 → col 2, etc

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 15 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 15 \end{bmatrix}$$

cofactor matrix $[A_{ij}]$ $\quad\quad$ A_{ij} : ij^{th} cofactor of A

$$A = \begin{bmatrix} 0^{+} & 1^{-} & 0^{+} \\ 4^{-} & 2^{+} & 5^{-} \\ 2^{+} & 0^{-} & 5^{+} \end{bmatrix}$$

$$A_{11} = + \begin{vmatrix} 2 & 5 \\ 0 & 5 \end{vmatrix} \qquad A_{12} = - \begin{vmatrix} 4 & 5 \\ 2 & 5 \end{vmatrix} \qquad A_{13} = \begin{vmatrix} 4 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= 10 \qquad\qquad = -10 \qquad\qquad = -4$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$[A_{ij}] = \begin{bmatrix} 10 & -10 & -4 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

Cramer's Rule → <u>Inverse Matrix Theorem</u>

$$A^{-1} = \frac{[A_{ij}]^T}{|A|}$$