3.6 Determinants (part 2)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \det A = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(6) = -2$$

$$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \qquad \det A^T = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2$$

$$A = \begin{bmatrix} 3 & 0 & 3 \\ 3 & 4 & 3 \\ 0 & 5 & -2 \end{bmatrix}$$

expand along any column or row

here, along column 1

$$\det A = (3)\begin{vmatrix} 4 & 3 \\ 5 & -2 \end{vmatrix} - (3)\begin{vmatrix} 0 & 3 \\ 5 & -2 \end{vmatrix} = (3)(-23) - (3)(-15)$$

$$A^T = \begin{bmatrix} 3 & 3 & 0 \\ 0 & 4 & 5 \\ 3 & 3 & -2 \end{bmatrix}$$

choose to expand along row 1

$$\det A^T = (3)\begin{vmatrix} 4 & 5 \\ 3 & -2 \end{vmatrix} - (3)\begin{vmatrix} 0 & 5 \\ 3 & -2 \end{vmatrix} =$$

same

so, $\boxed{\det A^T = \det A}$

triangular matrix : $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ upper triangular

$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ lower triangular

$$\det\left(\begin{bmatrix} \boxed{1} & 2 \\ 0 & \boxed{3} \end{bmatrix}\right) = (1)(3) - (0)(2) = \boxed{3}$$

$$A = \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & \boxed{4} & 5 \\ 0 & 0 & \boxed{6} \end{bmatrix}$$

along column 1

$$\det A = (1)\begin{vmatrix} 4 & 5 \\ 0 & 6 \end{vmatrix} = \boxed{(1)}\boxed{(4)}\boxed{(6)} = 24$$

> triangular or diagonal matrix : determinant is product of main diagonal elements

Gaussian elimination produces triangular matrix

Can we do elimination first, produce triangular matrix, then find det?

Yes, as long as we remember the following :

1. exchanging two rows switches sign of determinant

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\det A = -2$

$B = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ $\det B = 2$

2. multiplying one row by k $(k \neq 0)$ multiplies the determinant by the same k

$A = \begin{bmatrix} \frac{1}{10} & \frac{1}{5} \\ 3 & 4 \end{bmatrix}$ $\det A = \frac{2}{5} - \frac{3}{5} = -\frac{1}{5}$

$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ 10 times row 1 of A

$\det B = -2 = 10 \cdot \det A$

3. multiply one row and adding to another does NOT change the determinant

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \det A = -2$$

$$\xrightarrow{(-3)R_1 + R_2} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$ triangular, determinant is $(1)(-2) = -2$

example

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -3 & 2 \\ 5 & 1 & 9 \end{bmatrix}$$ "full matrix"

~~to~~ try to reduce to triangular (or at least introduce zeros) then find determinant

these do NOT affect determinant $\left\{ \xrightarrow[(2)R_3 + R_1]{(3)R_3 + R_2} \right.$

$$\begin{bmatrix} 12^{+} & 0^{-} & 21^{+} \\ 18^{-} & 0^{+} & 29^{-} \\ 5^{+} & 1^{-} & 9^{+} \end{bmatrix}$$ cofactor expansion along column 2

$$\det A = -(1) \begin{vmatrix} 12 & 21 \\ 18 & 29 \end{vmatrix} = -\left[(12)(29) - (18)(21)\right] = 30$$

Since $\det A^T = \det A$, the three rules above also apply to columns (elementary column operations)

example

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 3 & -3 & 2 \\ 5 & 1 & 9 \end{bmatrix}$$

$$\xrightarrow{C_2 + C_1} \begin{bmatrix} 0 & -2 & 3 \\ 0 & -3 & 2 \\ 6 & 1 & 9 \end{bmatrix}$$

does NOT change determinant

cofactor along column 1

$$\det A = (6)\begin{vmatrix} -2 & 3 \\ -3 & 2 \end{vmatrix} = (6)(-4+9) = 30$$

do NOT mix column and row operations

ALL column or ALL row ops

example

$$A = \begin{bmatrix} 2 & 4 & -2 & 6 \\ 1 & 2 & 5 & 4 \\ 1 & 1 & 2 & 4 \\ 0 & 2 & -6 & 3 \end{bmatrix}$$

$$\xrightarrow{\text{swap}(R_1, R_3)} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 5 & 4 \\ 2 & 4 & -2 & 6 \\ 0 & 2 & -6 & 3 \end{bmatrix}$$

determinant sign change

no effect on det(A)

$$\xrightarrow[(-2)R_1 + R_3]{(-1)R_1 + R_2} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & -6 & -2 \\ 0 & 2 & -6 & 3 \end{bmatrix}$$

$$\xrightarrow[(-2)R_2 + R_4]{(-2)R_2 + R_3} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -12 & -2 \\ 0 & 0 & -12 & 3 \end{bmatrix}$$

$$\xrightarrow{(-1)R_3+R_4} \underbrace{\begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -12 & -2 \\ 0 & 0 & 0 & 5 \end{bmatrix}}$$

determinant is $(1)(1)(-12)(5) = -60$

one swap earlier, so $\det A = -(-60) = 60$