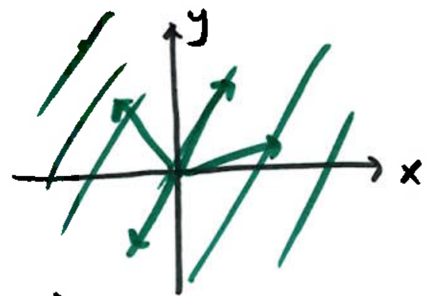
4.2 The Vector Space R^n and Subspaces

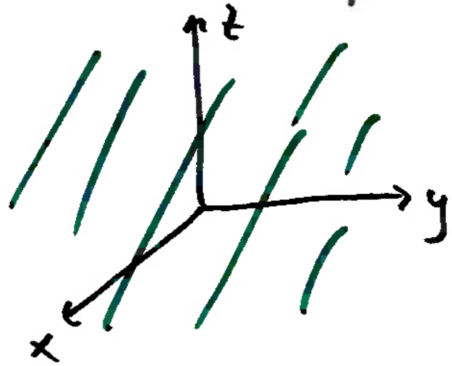
vector space R^n is the space containing all vectors with n components

R^1: all vectors $[x]$ x is real

R^2: $\begin{bmatrix} x \\ y \end{bmatrix}$ or (x, y) x, y reals



R^3: $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ x, y, z reals



R^4, R^5, etc same idea

inside the vector space R^n, we can multiply and add vectors as we normally do

all vector spaces have 8 fundamental properties

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$ (commutativity)

$\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$ (associativity)

$\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$ (existence of zero vector)

$\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$ (additive inverse)

$a(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$ (distributivity)

$(a+b)\vec{u} = a\vec{u} + b\vec{u}$

$a(b\vec{u}) = (ab)\vec{u}$

$1(\vec{u}) = \vec{u}$

ALL of these work in R^n vector space (n-component vectors)

vector space don't just have to contain "vectors"

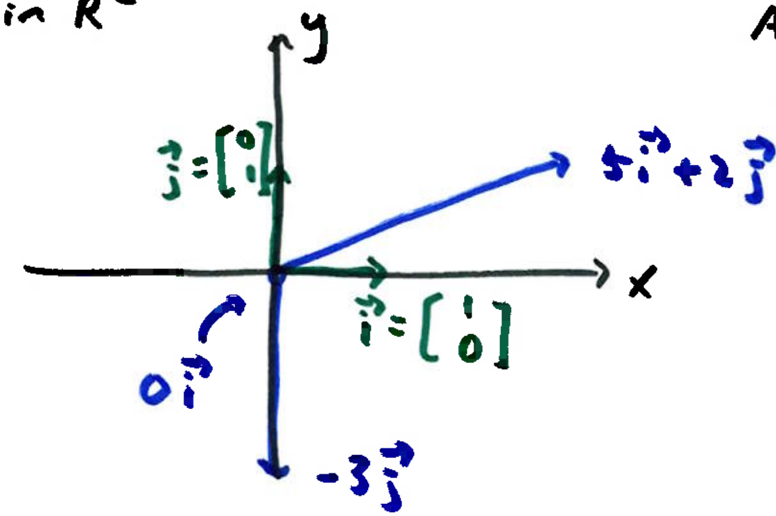
M_2 : vector space containing all 2x2 matrices
all 8 properties still hold

P_2 : all second degree polynomials

notice R^1 is contained in R^2, R^2 is in R^3, and so on.

also, notice doing operations on vectors in R^n the result stays in R^n

for example, in R^2



ANY linear combination of R^2 vectors stay in R^2, never go into R^3 (never gain a z component)

likewise, all vectors in R^1 stay in R^2 after any operation

but sometimes R^2 vectors are also vectors in R^1: for example, $5\vec{i} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$

Since R^1 is entirely contained in R^2, we say R^1 is a <u>subspace</u> of R^2

(R^2 is a subspace of R^3 and so on)

a subspace is a vector space, so has all 8 properties listed earlier

in addition, a subspace is <u>closed</u> under <u>addition</u> and <u>scalar multiplication</u>

closed under addition : pick any two vectors in subspace, their sum MUST remain in subspace

so, if $\vec{u}$ and $\vec{v}$ are in subspace W, then $\vec{u}+\vec{v}$ also stays in W

closed under scalar multiplication : if $\vec{u}$ is in W, then $c\vec{u}$ is also in W

for example, R^2 is a subspace of R^3 containing all vectors

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

pick $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ 0 \end{bmatrix}$ $\quad \vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ 0 \end{bmatrix}$ $\quad x_1, y_1, x_2, y_2$ reals

$$\vec{u}+\vec{v} = \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} x_3 \\ y_3 \\ 0 \end{bmatrix}}_{\text{also in } R^2} \qquad x_3, y_3 \text{ reals}$$

this shows closure under addition

$$c\vec{u} = \begin{bmatrix} cx_1 \\ cy_1 \\ 0 \end{bmatrix}$$ $\rightarrow$ still in R^2 so this shows closure under scalar multiplication

note the closure requirements imply $\vec{0}$ is in subspace

this is a necessary requirement for W to be a subspace

<u>example</u> W is a subset of R^3 such that $y=-1$. Is W a subspace?

vectors in W : $\begin{bmatrix} x \\ -1 \\ z \end{bmatrix}$ x, z reals

for W to be a subspace : closure under addition

closure under scalar multiplication

($\vec{0}$ is in W)

check closure under addition:

$\vec{u} = \begin{bmatrix} x_1 \\ -1 \\ z_1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} x_2 \\ -1 \\ z_2 \end{bmatrix}$ $\vec{u}+\vec{v} = \begin{bmatrix} x_1+x_2 \\ \boxed{-2} \\ z_1+z_2 \end{bmatrix}$ not in W so fails closure under addition

W is NOT a subspace

$c\vec{u} = \begin{bmatrix} cx_1 \\ -c \\ cz_1 \end{bmatrix}$ in W?

no, because in W only if $c=1$

a very important subspace of $\mathbb{R}^n$ is the space containing all solutions to $A\vec{x} = \vec{0}$

this space contains ALL $\vec{x}$ such that $A\vec{x} = \vec{0}$

<u>example</u>

$$\begin{aligned} x_1 - 4x_2 - 3x_3 - 7x_4 &= 0 \\ 2x_1 - x_2 + x_3 + 7x_4 &= 0 \\ x_1 + 2x_2 + 3x_3 + 11x_4 &= 0 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{bmatrix}}_{A} \underset{\vec{x}}{\vec{x}} = \underset{\vec{0}}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -4 & -3 & -7 & 0 \\ 2 & -1 & 1 & 7 & 0 \\ 1 & 2 & 3 & 11 & 0 \end{bmatrix}$$

$$\longrightarrow \cdots \longrightarrow \begin{bmatrix} \boxed{1} & 0 & 1 & 5 & 0 \\ 0 & \boxed{1} & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2 pivots, 4 variables, so $4-2=2$ free variables

choose $x_4 = t$ $\quad x_3 = s$

row 2: $\quad x_2 + x_3 + 3x_4 = 0 \rightarrow x_2 = -s-3t$

row 1: $\quad x_1 + x_3 + 5x_4 = 0 \rightarrow x_1 = -s-5t$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s-5t \\ -s-3t \\ s \\ t \end{bmatrix}$$

$$\underbrace{= s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$$

So the subspace $A\vec{x}=\vec{0}$

contains all linear combinations

$$\underbrace{\text{of } \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}}$$

two-dimensional space because we use two vectors to form linear combos