1.2 Integrals as General and Particular Solutions

first-order DE: $\dfrac{dy}{dx} = \underbrace{f(x,y)}_{\text{can depend on } x \text{ and } y}$

for example, $\dfrac{dy}{dx} = xy^2$

a very simple type: $\dfrac{dy}{dx} = f(x)$ only independent variable on right side

for example, $\dfrac{dy}{dx} = 2x \rightarrow$ solution is $y = \int 2x\,dx$

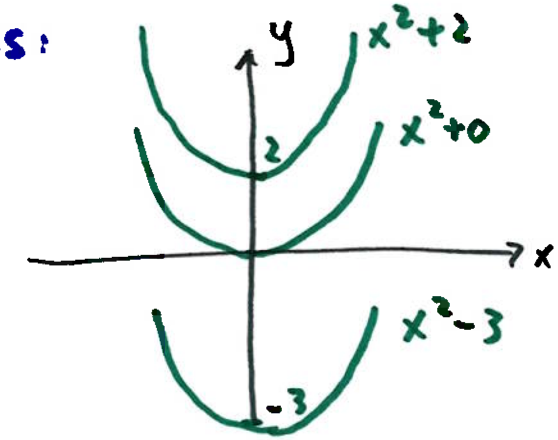
$$\boxed{y = x^2 + c}$$ solution!

check: is $\dfrac{dy}{dx} = 2x$?

$y' = \dfrac{d}{dx}(x^2 + c) = 2x$

yes.

solution curves:



infinitely many solutions

we call the solution $y = x^2 + c$

the <u>general solution</u>

if we specify a point that the solution curve goes through

for example, $\underbrace{y(0) = 2}_{\text{initial condition}}$

$y(x) = x^2 + c$ general solution

$2 = (0)^2 + c$

$c = 2$

so, $\boxed{y = x^2 + 2}$ <u>particular solution</u>

another example $\frac{dy}{dx} = x\sqrt{x^2+1}$ $\quad y(0)=2$

integrate: $y = \int x\sqrt{x^2+1}\, dx$

$u = x^2+1 \quad du = 2x\,dx$

$$y = \int \frac{1}{2}u^{1/2}\,du = \frac{1}{2}\cdot\frac{2}{3}u^{3/2} + C$$

$$\boxed{y = \frac{1}{3}(x^2+1)^{3/2} + C}$$ general solution

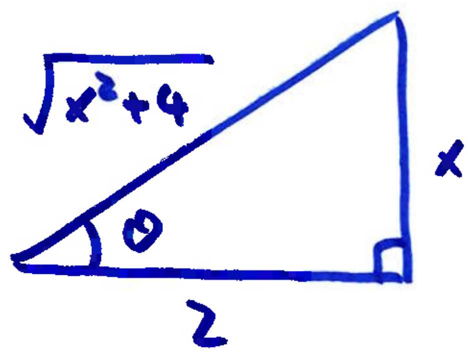
use $y(0)=2$ to find C

$$2 = \frac{1}{3}(1)^{3/2} + C \qquad C = 2 - \frac{1}{3} = \frac{5}{3}$$

so, $$\boxed{y = \frac{1}{3}(x^2+1)^{3/2} + \frac{5}{3}}$$ particular solution example: $\frac{dy}{dx} = \frac{1}{x^2+4}$ $\quad y(0)=10$

trig subs: $\frac{dy}{dx} = \frac{1}{(\sqrt{x^2+4})^2}$

triangle with sides: $\sqrt{x^2+4}$, x, 2



$\tan\theta = \frac{x}{2}$

$x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$

$$y = \int \frac{1}{x^2+4}\,dx = \int \frac{1}{4(\tan^2\theta+1)} \cdot 2\sec^2\theta \, d\theta$$

$$= \int \frac{1}{4\,\cancel{\sec^2\theta}} \cdot 2\,\cancel{\sec^2\theta} \, d\theta$$

$$= \int \frac{1}{2}\, d\theta = \frac{1}{2}\theta + C$$

$\tan\theta = \frac{x}{2}$ $\theta = \tan^{-1}\left(\frac{x}{2}\right)$

$$\boxed{y = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C}$$

General solution

$y(0) = 10$ $10 = \frac{1}{2}\tan^{-1}(0) + C = C$

$$\boxed{y = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + 10}$$

Same if the order is higher

$$\frac{d^2y}{dx^2} = f(x)$$

$$\frac{dy}{dx} = \int f(x)\,dx + C_1 = F(x) + C_1$$

C_1: first integration constant

need $y'(x_0) = y_0'$

$$y = \int (F(x) + C_1)\,dx = \int F(x)\,dx + C_1 x + C_2$$

C_2: second constant

need $y(x_0) = y_0$

2nd-order appears frequently in applications: $F = ma$ Newton's 2nd Law

if $s(t)$ is position and $v(t)$ is velocity and $a(t)$ is acceleration

then $a = \frac{F}{m} = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

for example, $a(t) = -10$, $v(0) = 20$, $s(0) = 5$

$\underbrace{v(0)=20}_{\text{initial velocity}}$ $\underbrace{s(0)=5}_{\text{initial position}}$

find $s(t)$

$$v(t) = \int a(t)\,dt = \int -10\,dt = -10t + C_1$$

usually better to find C_1 before next integral

$v(t) = -10t + C_1$

$v(0) = 20$

$20 = 0 + C_1$

$$\boxed{v(t) = -10t + 20}$$

$$s(t) = \int v(t)\,dt = \int (-10t + 20)\,dt = -5t^2 + 20t + C_2$$

$$s(0) = 5$$

$$5 = 0 + 0 + C_2$$

$$\boxed{s(t) = -5t^2 + 20t + 5}$$